PISIER'S $K-$CONVEXITY INEQUALITY: AN EXPOSITION

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We aim to show the bound $K(X) \lesssim \log(1 + d_{BM}(X, \ell_2^m))$ on the $K-$convexity constant $K(X)$ of an $m-$dimensional Banach space $(X, \|\cdot\|_X)$.

1. Preliminaries

Rademacher Projection. Consider the cube $E^n = \{\pm 1\}^n$ with the uniform measure μ. Then $L_2(E^n, \mu)$ is a Hilbert space whose orthonormal basis is the set $\{w_A \mid A \subseteq n\}$ of Walsch functions as described next. The Rademacher functions $r_i : E^n \to \{\pm 1\}$ are given by $r_i(\boldsymbol{\varepsilon}) = r_i(\varepsilon_1, \cdots, \varepsilon_n) := \varepsilon_i$. For any subset $A \subseteq [n]$ define the Walsch function $w_A : E^n \to \{\pm 1\}$ as $w_A(\boldsymbol{\varepsilon}) := \prod_{i \in A} r_i(\boldsymbol{\varepsilon})$ with the agreement that $w_\varnothing \equiv 1$. Any function $f : E^n \to X$ (with $(X, \|\cdot\|_X)$ a Banach space) can be written as $f(\boldsymbol{x}) = \sum_{A \subseteq [n]} \hat{f}(A) w_A(\boldsymbol{x}) \, \forall \, \boldsymbol{x} \in E^n$ where the Fourier coefficients $\hat{f}(A) \in X$ are determined by $\hat{f}(A) = \sum_{\boldsymbol{\varepsilon} \in E^n} w_A(\boldsymbol{\varepsilon}) f(\boldsymbol{\varepsilon}) \mu(\boldsymbol{\varepsilon}) = \frac{1}{2^n} \sum_{\boldsymbol{\varepsilon} \in E^n} w_A(\boldsymbol{\varepsilon}) f(\boldsymbol{\varepsilon})$. The Rademacher projection is the operator $Rad_n : L_2(E^n; X) \to L_2(E^n; X)$ defined by $Rad_n(f) := \left(\boldsymbol{x} \mapsto \sum_{i \in [n]} \hat{f}\left(\{i\}\right) w_{\{i\}}(\boldsymbol{x}) = \sum_{i \in [n]} \hat{f}\left(\{i\}\right) x_i \right)$. Note that $L_2(E^n; X)$ is the space of all $X-$valued functions on E^n with the norm $\|f\|_{L_2(E^n;X)} = \sqrt{\frac{1}{2^n} \sum_{\boldsymbol{x} \in E^n} f(\boldsymbol{x})}$. The $K-$convexity constant of X is

$$K(X) = \sup_n \|Rad_n\|_{L_2(E^n;X) \to L_2(E^n;X)}$$

and we say X is $K-$convex if $K(X) < \infty$.

Convolutions. We will consider E^n as a group, for Fourier analysis, with coordinate-wise multiplication. That is, if $\boldsymbol{x} = (x_1, \cdots, x_n), \boldsymbol{y} = (y_1, \cdots, y_n) \in E^n$ then $\boldsymbol{xy} = (x_1 y_1, \cdots, x_n, y_n)$. The identity is simply $\mathbf{1}$, the all-ones vector, and the inverse of $\boldsymbol{x} \in E^n$ is itself. Convolution is a powerful tool when it comes to Fourier analysis. Fix a Banach space $(X, \|\cdot\|)$. The convolution of $f : E^n \to X$ and $h : E^n \to \mathbb{R}$ is given by

$$f * h(\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{\varepsilon}} \left[f(\boldsymbol{x\varepsilon}) h(\boldsymbol{\varepsilon}) \right] = \frac{1}{2^n} \sum_{\boldsymbol{\varepsilon} \in E^n} f(\boldsymbol{x\varepsilon}) h(\boldsymbol{\varepsilon}).$$

Date: April 2025.

Note that the expectation is taken with respect to the uniform measure on the boolean cube. The fact that $\boldsymbol{x}\boldsymbol{\varepsilon} \overset{D}{=} \boldsymbol{\varepsilon}$, for a fixed $\boldsymbol{x} \in E^n$, allows us to write $f * h(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{\varepsilon}}\left[f(\boldsymbol{\varepsilon})h(\boldsymbol{x}\boldsymbol{\varepsilon})\right]$. The intuition that Rad_n is the 'linear' part is captured by the fact that $Rad_n f = f * \ell$ where $\ell \coloneqq \sum_{i=1}^{n} w_{\{i\}}$. Indeed, $f * \ell(\boldsymbol{x}) = \sum_{i\in[n]} \underbrace{\sum_{\boldsymbol{\varepsilon}\in E^n} \frac{f(\boldsymbol{\varepsilon})\varepsilon_i}{2^n}}_{\hat{f}(\{i\})} x_i = \sum_{i\in[n]} \hat{f}(\{i\})x_i = Rad_n(f)(\boldsymbol{x})$.

An interesting way that that Fourier analysis interacts with convolutions is that it is the natural way for preserving multiplicative structure, in the sense that $f * g(\boldsymbol{x}) = \sum_{A\subseteq n} \hat{f}(A)\hat{g}(A)w_A(\boldsymbol{x})$. In other words, $\widehat{f * g}(A) = \hat{f}(A)\hat{g}(A)$. This is due to the orthonormality of the Walsch characters. We will also use the following fact about 'sub-multiplicativity of norms' under convolutions, which has been proved in Appendix A.1.

Lemma 1. *Let $f : E^n \to X, h : E^n \to \mathbb{R}$. Then*

$$\|f * h\|_{L_2(E^n;X)} \leq \|f\|_{L_2(E^n;X)} \|h\|_{L_1(E^n)}.$$

A highly nonlinear approximation to ℓ. Recall the 'almost moving delta mass' with real parameter t: $\alpha(\boldsymbol{x}) = \alpha(t, \boldsymbol{x}) \coloneqq \prod_{i=1}^{n}(1 + tx_i) = \sum_{A\subseteq[n]} t^{|A|}w_A(\boldsymbol{x})$. The large powers of t works in our favour to help ignore the terms larger than 1. However, we will play around a bit with α to get precisely what we need. Consider the function $\varphi_r(\theta) \coloneqq \frac{2r-1}{r}\frac{\sin(r\theta)}{\sin^2\theta}$ on $[0, 2\pi]$. Consider $\Gamma_r \coloneqq \left\{\frac{k\cdot 2\pi}{4r} \;\middle|\; k \in [0, 4r-1] \cap \mathbb{Z}\right\}$ and $\Delta_r = \Gamma_r \setminus \{0, \pi\}$ each equipped with the uniform measure, where r is an odd integer parameter to be decided later. As well as the function $g_r(\boldsymbol{x}) = \mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\alpha\left(\frac{\sin\theta}{2}, \boldsymbol{x}\right)\right]$. Expanding we get

$$g_r(\boldsymbol{x}) = 2\mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\alpha\left(\frac{\sin\theta}{2}, \boldsymbol{x}\right)\right] = 2\sum_{A\subseteq[n]} \mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\left(\frac{\sin\theta}{2}\right)^{|A|}\right] w_A(\boldsymbol{x}).$$

So $\hat{g}_r(A) = 2\mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\left(\frac{\sin\theta}{2}\right)^{|A|}\right]$.

The following facts, proven in Appendix A.2, help showing that g linearly approximates g_1:

Proposition 2. *(a) $\mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\sin^k\theta\right] = \delta_{k,1}$ for $0 \leq k \leq r$.*

(b) $\mathbb{E}_{\theta\sim\Delta_r}\left[|\varphi_r(\theta)|\right] \leq 4r$.

Corollary 3. *$\hat{g}_r(A) = \delta_{k,1}$ if $0 \leq |A| \leq r$ and $|\hat{g}_r(A)| \leq \frac{4r}{2^r}$ if $|A| > r$.*

Proof. The equality case directly follows from Proposition 2(a).
Now say $|A| > r$. Then $|\hat{g}_r(A)| \leq \frac{2}{2^{|A|}}\mathbb{E}\left[|\varphi_r(\theta)|\right] \leq \frac{2}{2^{r+1}} \cdot 4r = \frac{4r}{2^r}$ by Proposition 2(b). ■

Corollary 4. $\|g_r\|_{L_1(E^n)} = \mathbb{E}_{\boldsymbol{\varepsilon}}\left[|g_r(\boldsymbol{\varepsilon})|\right] \le 8r$.

Proof.

$$\begin{aligned}
\frac{1}{2}\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|g_r(\boldsymbol{\varepsilon})|\right] &\le \mathbb{E}_{\boldsymbol{\varepsilon}}\mathbb{E}_{\theta\sim\Delta_r}\left[\left|\varphi_r(\theta)\prod_{i=1}^{n}\left(1+\frac{\sin\theta}{2}\varepsilon_i\right)\right|\right] \\
&= \mathbb{E}_{\theta\sim\Delta_r}\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|\varphi_r(\theta)|\prod_{i=1}^{n}\left(1+\frac{\sin\theta}{2}\varepsilon_i\right)\right] && \because 1+\frac{\sin\theta}{2}\varepsilon_i > 0 \\
&= \mathbb{E}_{\theta\sim\Delta_r}\left[|\varphi_r(\theta)|\prod_{i=1}^{n}\mathbb{E}_{\boldsymbol{\varepsilon}}\left(1+\frac{\sin\theta}{2}\varepsilon_i\right)\right] \\
&= \mathbb{E}_{\theta\sim\Delta_r}\left[|\varphi_r(\theta)|\right] \overset{\text{Proposition } 2(b)}{\le} 4r.
\end{aligned}$$

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Banach-Mazur distance.

Definition 5. *If E, F are isomorphic Banach spaces, their Banach-Mazur distance is*

$$d(E,F) = d_{BM}(E,F) := \inf\left\{\|T\|\cdot\left\|T^{-1}\right\| \;\middle|\; T : E \overset{\sim}{\to} F\right\}.$$

For finite dimensional Banach spaces X, we want to look at their distances[1] from a Hilbert space H of the same dimension which will be used for bounding norms of convolutions. We will be specially interested in $H = \ell_2^m$ where $m := \dim X$.

Let $f : E^n \to X, h : E^n \to \mathbb{R}$. Consider the Fourier representation $h = \sum_A \hat{h}(A)w_A$. Fix any Hilbert space H isomorphic to X, and let $D := d_{BM}(X,H)$. For any $\varepsilon > 0, \exists\, T = T_\varepsilon : X \overset{\sim}{\to} H$ such that $\|T\|\,\|T^{-1}\| \le (1+\varepsilon)D$. Note that $f * h = \sum_A \hat{f}(A)\hat{h}(A)w_A$ with $\hat{f}(A) \in X, \hat{h}(A) \in \mathbb{R}$ and $w_A : E^n \to \{\pm 1\}$ are the Walsch functions, $\forall\, A \subseteq [n]$. So $T(f*h) = \sum_A T(\hat{f}_A)\hat{h}(A)w_A$. We thus have

$$\begin{aligned}
\|f*h\|_{L_2(E^n;X)} &= \left\|T^{-1}\circ T\circ(f*h)\right\|_{L_2(E^n;X)} \\
&\le \left\|T^{-1}\right\|\,\|T(f*h)\|_{L_2(E^n;H)} \\
&= \left\|T^{-1}\right\|\sqrt{\sum_A \left\|\hat{h}(A)T(\hat{f}(A))\right\|_H^2} \\
&= \left\|T^{-1}\right\|\sqrt{\sum_A \left|\hat{h}(A)\right|\left\|T(\hat{f}(A))\right\|_H^2}
\end{aligned}$$

[1]warning: d_{BM} is not an honest metric

$$\leq \left\|T^{-1}\right\| \left(\max_{A\subseteq[n]} \left|\hat{h}(A)\right|\right) \sqrt{\sum_A \left\|T(\hat{f}(A))\right\|_H^2}$$

$$= \left(\max_{A\subseteq[n]} \left|\hat{h}(A)\right|\right) \left\|T^{-1}\right\| \left\|T(f)\right\|_{L_2(E_2^n;H)}$$

$$\leq \left(\max_{A\subseteq[n]} \left|\hat{h}(A)\right|\right) \left\|T^{-1}\right\| \left\|T\right\| \left\|f\right\|_{L_2(E_2^n;X)}$$

$$\leq (1+\varepsilon) D \left(\max_{A\subseteq[n]} \left|\hat{h}(A)\right|\right) \left\|f\right\|_{L_2(E_2^n;X)}.$$

Since this is true $\forall\, \varepsilon > 0$, we conclude that

Lemma 6. *Let $f : E^n \to X, h : E^n \to \mathbb{R}$, where X is $m-$dimensional, and consider the Fourier representation $h = \sum_A \hat{h}(A) w_A$. Then*

$$\left\|f * h\right\|_{L_2(E^n;X)} \leq d(X, \ell_2^m) \left(\max_{A\subseteq[n]} \left|\hat{h}(A)\right|\right) \left\|f\right\|_{L_2(E_2^n;X)}.$$

2. The Final Proof

We have established $g_r = \sum_A \hat{g}(A) w_A = \sum_{i\in[n]} w_{\{i\}} + \sum_{|A|>r} \hat{g}(A) w_A$ with $|\hat{g}(A)| \leq \frac{4r}{2^r}$ (small) for $|A| > r$. Letting $\psi_r \coloneqq \sum_{|A|>r} \hat{g}(A) w_A$, we have $g_r = \ell + \psi_r$. We note that $\hat{\psi}_r(A) = \begin{cases} 0 & \text{if } |A| \leq r \\ \hat{g}(A) & \text{if } r < |A| \leq n \end{cases}$. Thus, g_r and ℓ are 'close' functions. Now let's start bounding $\left\|f * \ell\right\|_{L_2(E^n;X)}$. Let $D \coloneqq d(X, \ell_2^{\dim X})$.

Firstly note that

$$\begin{aligned}
\left\|f * \ell\right\|_{L_2(E^n;X)} &= \left\|f * (g_r - \psi_r)\right\|_{L_2(E^n;X)} \\
&\leq \left\|f * g_r\right\|_{L_2(E^n;X)} + \left\|f * \psi_r\right\|_{L_2(E^n;X)} \\
&\leq \underbrace{\left\|f\right\|_{L_2(E^n;X)} \left\|g_r\right\|_{L_1(E^n)}}_{\text{[Lemma 1]}} + \underbrace{D \left(\max_{A\subseteq[n]} \left|\hat{\psi}_r(A)\right|\right) \left\|f\right\|_{L_2(E_2^n;X)}}_{\text{[Lemma 6]}} \\
&= \left(\left\|g_r\right\|_{L_1(E^n)} + D \left(\max_{A\subseteq[n]} \left|\hat{\psi}_r(A)\right|\right)\right) \left\|f\right\|_{L_2(E_2^n;X)} \\
&\leq \left(\underbrace{8r}_{\text{[Corollary 3]}} + D \cdot \underbrace{4r \cdot 2^{-r}}_{\text{[Corollary 3]}}\right) \left\|f\right\|_{L_2(E_2^n;X)} \\
&= 8r \left(1 + D \cdot 2^{-(r+1)}\right) \left\|f\right\|_{L_2(E_2^n;X)}.
\end{aligned}$$

To complete the proof, choose r to be an odd number such that $2^{r-1} \leq D+1 \leq 2^r$ so that $8r(1+D\cdot 2^{-(1+r)}) \leq 8\left(\lg(1+D)+1\right)\left(1+\frac{D}{2(D+1)}\right) \lesssim \lg(1+D)$ thus proving that $\|Rad_n\|_{L_2(E^n;X)\to L_2(E^n;X)} \lesssim \lg(1+D)\ \forall\, n$. This means $K(X) \lesssim \lg(1+d(X,\ell_2^{\dim X}))$.

To get a more quantitative bound, we can use John's theorem which states that $d(X,\ell_2^{\dim X}) \leq \sqrt{\dim X}$, thus giving $K(X) \lesssim \log\dim X$.

Appendix A.

A.1. **Proof of Lemma 1.** We want to show $\|f*h\|_{L_2(E^n;X)} \le \|f\|_{L_2(E^n;X)} \|h\|_{L_1(E^n)}$.

$$\begin{aligned}
\|f*h\|^2_{L_2(E^n;X)} &= \int_{E^n} \|f*h(\boldsymbol{x})\|_X^2 \,\mathrm{d}\boldsymbol{x} \\
&= \int_{E^n} \|\mathbb{E}_{\boldsymbol{\varepsilon}}\left[h(\boldsymbol{\varepsilon})f(\boldsymbol{x}\boldsymbol{\varepsilon})\right]\|_X^2 \,\mathrm{d}\boldsymbol{x} \\
&= \int_{E^n} \left(\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\,\|f(\boldsymbol{x}\boldsymbol{\varepsilon})\|_X\right]\right)^2 \,\mathrm{d}\boldsymbol{x} \\
&= \int_{E^n} \left(\mathbb{E}_{\boldsymbol{\varepsilon}}\left[\sqrt{|h(\boldsymbol{\varepsilon})|}\cdot\sqrt{|h(\boldsymbol{\varepsilon})|}\,\|f(\boldsymbol{x}\boldsymbol{\varepsilon})\|_X\right]\right)^2 \,\mathrm{d}\boldsymbol{x} \\
&\overset{\text{Cauchy-Schwarz}}{\le} \int_{E^n} \mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\right]\cdot\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\,\|f(\boldsymbol{x}\boldsymbol{\varepsilon})\|_X^2\right] \mathrm{d}\boldsymbol{x} \\
&= \mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\right]\cdot\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\int_{E^n}\|f(\boldsymbol{x}\boldsymbol{\varepsilon})\|_X^2\,\mathrm{d}\boldsymbol{x}\right] \\
&= \mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\right]\cdot\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\int_{E^n}\|f(\boldsymbol{x})\|_X^2\,\mathrm{d}\boldsymbol{x}\right] \\
&= \left(\mathbb{E}_{\boldsymbol{\varepsilon}}\left[|h(\boldsymbol{\varepsilon})|\right]\right)^2\int_{E^n}\|f(\boldsymbol{x})\|_X^2\,\mathrm{d}\boldsymbol{x} \\
&= \|h\|^2_{L_1(E^n)}\,\|f\|^2_{L_2(E^2;X)}\,.
\end{aligned}$$

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A.2. **Proof of Proposition 2.**

(a) We first want to show $\mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\sin^k\theta\right] = \delta_{k,1}$ for $0 \le k \le r$.

- Say $k = 0$. $\varphi(\theta) = -\varphi(2\pi - \theta)$ and since the expectation is taken with respect to the uniform measure and for every $\theta \in \Delta_r$, we also have $2\pi - \theta \in \Delta_r$, the statement is true for $k = 0$.
- Say $k = 1$. We will show $\mathbb{E}_{\theta\sim\Delta_r}\left[\varphi_r(\theta)\sin\theta\right] = 1$ by inducting on odd r. For now, just consider the sum

$$\begin{aligned}
\sum_{\theta\in\Delta_r}\frac{\sin(r\theta)}{\sin\theta} &= \sum_{\theta\in\Delta_r}\frac{e^{ir\theta}-e^{-ir\theta}}{e^{i\theta}-e^{-i\theta}} = \sum_{\theta\in\Delta_r}\sum_{j=0}^{r-1} e^{ij\theta}\cdot e^{-i(r-1-j)\theta} \\
&= \sum_{\theta\in\Delta_r}\sum_{j=0}^{r-1} e^{i(2j-r+1)\theta} = \sum_{j=0}^{r-1}\sum_{\theta\in\Delta_r} e^{i(2j-r+1)\theta} \\
&= \sum_{j=0}^{r-1}\left(\sum_{\theta\in\Gamma_r} e^{i(2j-r+1)\theta} - 1 - \cos((\underbrace{2j-r+1}_{\text{even}})\pi)\right)
\end{aligned}$$

$$= \sum_{j=0}^{r-1} \left(\sum_{\theta \in \Gamma_r} e^{i(2j-r+1)\theta} - 2 \right) = \sum_{j=0}^{r-1} \sum_{\theta \in \Gamma_r} e^{i(2j-r+1)\theta} - 2r.$$

Recall that for any integer x, $\sum_{\theta \in \Gamma_r} e^{ix\theta} = 4r \cdot \mathbf{1}[x \equiv 0 \pmod{4r}]$ because Γ_r divides 2π into $4r$ equally spaced angles, whence the above sum becomes $4r - 2r = 2r$. Thus, $\mathbb{E}_{\theta \sim \Delta_r}[\varphi_r(\theta) \sin\theta] = \dfrac{1}{4r-2} \cdot \dfrac{2r-1}{r} \sum_{\theta \in \Delta_r} \dfrac{\sin(r\theta)}{\sin\theta} = 1.$

- For $k \geq 2$ we consider $q = k - 2 \in [0, r-2] \cap \mathbb{Z}$ and are interested in $\mathbb{E}_{\theta \sim \Delta_r}\left[\varphi_r(\theta) \sin^k \theta\right] = \dfrac{1}{2r} \sum_{\theta \in \Delta_r} \sin(r\theta) \sin^q \theta = \dfrac{1}{2r} \sum_{\theta \in \Gamma_r} \sin(r\theta) \sin^q \theta$. Let's instead look at $\sum_{\theta \in \Gamma_r} \sin(r\theta) \sin^q \theta = \sum_{\theta \in \Gamma_r} \left(\dfrac{e^{ir\theta} - e^{-ir\theta}}{2i} \right) \left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i} \right)^q$. For each $\theta \in \Gamma_r$, a typical term in the expansion of the power looks like (upto constants in $\mathbb{C}$) $e^{i\theta(2j-q)}$ for $j = 0, 1, \cdots, q$. Combining with the first bracket gives a typical term like $e^{i\theta(2j-q\pm r)}$. $2j - q \pm r$ is always in $[-2r, 2r]$ so all terms on expansion, after $\sum_{\theta \in \Gamma_r}$ are 0, unless $2j - q \pm r = 0$ for some $0 \leq j \leq q \leq r-2$. If $2j - q - r = 0$ then $0 \geq 2j - 2q = r + q > 0$, a contradiction. If $2j - q + r = 0$ then $0 \leq j = q - r < 0$, a contradiction again. Hence all terms have their exponent nonzero modulo $4r$, hence zero.

(b) Next we come to showing $\mathbb{E}_{\theta \sim \Delta_r}[|\varphi_r(\theta)|] \leq 4r$. Recall that $\varphi_r(\theta) = \frac{2r-1}{r} \frac{\sin(r\theta)}{\sin^2\theta}$ where θ takes the values $\frac{2\pi k}{4r}$ for $k \in [4r-1] \setminus \{2r\}$. By the symmetry of θ in all four coordinates to account for the fact that we only want to look at $\left|\sin^2\theta\right|$, we have

$$\mathbb{E}_{\theta \sim \Delta_r}[|\varphi_r(\theta)|] \leq \frac{4}{4r-2} \cdot \frac{2r-1}{r} \sum_{k=1}^{r} \frac{1}{\sin^2\left(\frac{2\pi k}{4r}\right)} \leq \frac{2}{r} \sum_{k=1}^{r} \frac{r^2}{j^2} \leq 2r \cdot \frac{\pi^2}{6} \leq 4r$$

where we used the inequality $\sin t \geq \frac{2}{\pi} t$ for $0 \leq t \leq \frac{\pi}{2}$.