Definition:

① A quiver is a (connected finite) directed graph.

② A representation of a quiver $Q = (V, \mathcal{E})$ is an assignment to each $i \in V$ a (finite dim'l) vector space V_i and to each edge $\alpha : i \to j$ a linear transformation $T_e : V_i \longrightarrow V_j$. $\boxed{i = s(e),\ j = t(e)}$

③ A subrepresentation of a representation $((V_i), (T_e))$ of quiver Q is a representation $((W_i), (L_e))$ s.t.

- $W_i \subseteq V_i \ \forall i$
- $T_e(W_{s(e)}) \subseteq W_{t(e)} \ \forall$ edges e
- $L_e = T_e|_{W_{s(e)}} \ \forall$ edges e.

④ A direct sum of reps $((V_i), (T_e))$ and $((W_i), (S_e))$ is just $((V_i \oplus W_i), (T_e \oplus S_e))$.

⑤ A homomorphism of two reps $((V_i), (T_e)) \longrightarrow ((W_i), (S_e))$ of a quiver Q is a collection of linear maps $\varphi_i : V_i \to W_i$ s.t.

$$\begin{tikzcd} V_i \arrow[r, "\varphi_i"] \arrow[d, "T_e"'] \arrow[dr, phantom, "\circlearrowleft"] & W_i \arrow[d, "S_e"] \\ V_j \arrow[r, "\varphi_j"'] & W_j \end{tikzcd}$$

commutes $\forall$ edges $e : i \to j$

⑥ A rep is said to be indecomposable if it is not the direct sum of two other representations.

Care only about indecomposables and not simples. A couple of reasons:

1) If there are no directed cycles in the quiver Q then the only simples are T_i (k at i, 0 elsewhere, all maps are 0)

2) Representations are not semisimple in general. For example, $k \longrightarrow k$ is not simple $\because$ $0 \longrightarrow k$ is a subrep. But it is not a sum of subrepresentations.

3) Thm (Azumaya, Fitting, Krull, Remak, Schmidt):
- V indecomposable $\Leftrightarrow$ $\mathrm{End}(V)$ is local (i.e. non-invertible elements form an ideal).
- Each representation decomposes into a finite direct sum of indecomposable representations, determined uniquely upto isom & perm.

Examples.

① Finding representations of $\bullet \circlearrowleft$ is same as finding vector space V with linear map $T: V \longrightarrow V$.

$V \circlearrowleft T$ and $W \circlearrowleft S$ are isom iff $V \simeq W$ (as v.s.) and $BTB^{-1} = S$ for some B.

So indecomposables given by JCF.

② K_r: $\bullet \overset{\vdots\, r}{\rightrightarrows} \bullet$ Reps are $V \underset{T_r}{\overset{T_1}{\rightrightarrows}} W$ whose classes are determined by simultaneous multiplication of something to left of $T_1, \dots, T_r$ right of $T_1, \dots, T_r$.

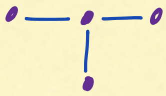
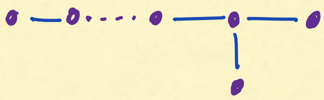
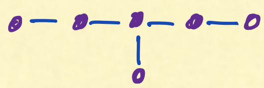
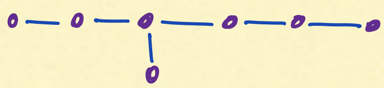
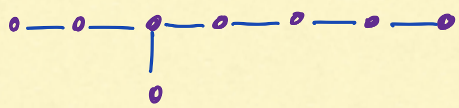
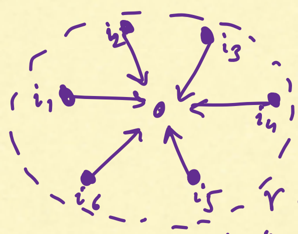
For $r \geq 2$ there are infinitely many non-isom representations (involved proof).

③ K_1: $\bullet \longrightarrow \bullet$ A_2

$$V \xrightarrow{T} W \simeq \ker T \longrightarrow 0 \oplus \frac{V}{\ker T} \longrightarrow \mathrm{Im}\, T \oplus 0 \longrightarrow \mathrm{coker}\, T$$

$$\simeq (\dim \ker T) \cdot (k \longrightarrow 0) \oplus (\dim(\mathrm{Im}\, T)) \cdot (k \longrightarrow k) \oplus (\dim \mathrm{coker}\, T)\, (0 \longrightarrow k).$$

3 indecomposables: $k \to 0$, $k \to k$, $0 \to k$.

List:	Diagram	No. of indecomposables.
A_2		3
A_3		6
A_4		10
D_4		12
(A) A_n		$\frac{n(n+1)}{2}$
(D) D_n		$n(n-1)$
(E) E_6		36
E_7		63
E_8		120
S_r: (Star)	 r of them, $r \geq 4$	$\infty \rightsquigarrow \left(\begin{tikzcd} & k \arrow[d, "(0,1)"] & \\ k \arrow[r, "(1,0)"'] & k^2 & k \arrow[l, "(1,1)"'] \\ & k \arrow[u, "(1,\lambda)"'] & \end{tikzcd} \right)$
K_2		$\infty \rightsquigarrow \left(k \overset{x_1}{\underset{x_2}{\rightrightarrows}} k \quad (x_1 : x_2) \in \mathbb{P}^1(k) \right)$

Def: A Quiver Q is said to be of <u>finite type</u> if it has only finitely many indecomposable representations

Thm (Gabriel). TFAE

① A quiver $Q = (\mathcal{V}, \mathcal{E})$ is of finite type

② The quadratic form defined on $\mathbb{R}^{\mathcal{V}}$ defined by

$$B_Q(\alpha) := \sum_{v \in \mathcal{V}} \alpha(v)^2 - \sum_{e \in \mathcal{E}} \alpha(s(e)) \cdot \alpha(t(e))$$

$\alpha(s(e))\,\alpha(t(e)) =: F(e)$

is +ve-definite.

③ The underlying graph of Q (forget orientations) of edges is A, D, or E.

Proof of (2) ⇒ (3).

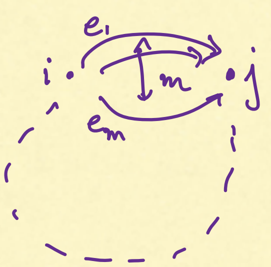
Claim: Q has no self loops.

Pf: 

Take $\alpha \in \mathbb{R}^{\mathcal{V}}$ s.t. $\alpha(i) = 1$ & $\alpha(v) = 0 \ \forall v \neq i$

So $B(\alpha) = \underset{\alpha_i^2}{\underset{\uparrow}{1}} - \underset{F(x)}{\underset{\uparrow}{1}} = 0$

Claim: No multiple edges present b/w two vertices.

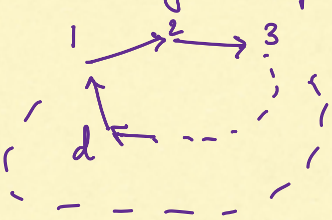
Pf: 

Take $\alpha = \begin{cases} j, i \mapsto 1 \\ l \mapsto 0 & l \neq i, j \end{cases}$

Then $B(\alpha) = 2 - F(e_1) - \cdots - F(e_m)$

$= 2 - m > 0 \Rightarrow m \leq 1$

Claim: No cycles present.

Pf: 

$\alpha(1) = \cdots = \alpha(d) = 1$

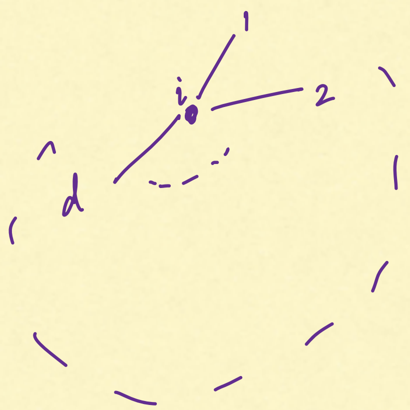
$\alpha(i) = 0$ if $i \neq 1, \ldots, d$

Then $B(\alpha) = d - F(1 \to 2) - F(2 \to 3) \cdots - F(d \to 1)$

$= d - d = 0$

Claim: Every vertex has deg ≤ 3.

Pf:



$\alpha(i) = 2, \quad \alpha(1) = \cdots = \alpha(d) = 1$

$\alpha(j) = 0$ if $j \neq 1, \ldots, d, i$

$B(\alpha) = d + 2 \quad - 2\big(F(i \to 1) + \cdots + F(i \to d)\big)$

$= d + 4 - 2d = 4 - d > 0$

$\Rightarrow \quad d \leq 3$

Claim: There is at most one vertex with 3 neighbours.

Pf

$\alpha(j_1) = \cdots = \alpha(j_d) = 2$

$\alpha(1) = \alpha(2) = \alpha(3) = \alpha(4) = 1$

$\alpha(\text{others}) = 0$

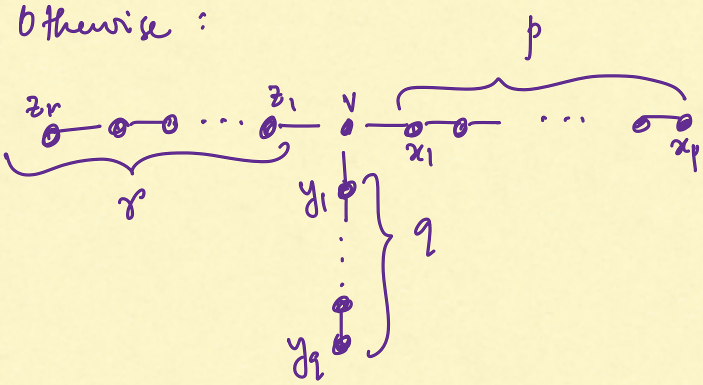
Then $B(\alpha) = \quad 4d - F(j_1 - 1) - F(j_1 - 2) - F(j_2 - 3) - F(j_2 - 4)$

$- F(j_1 - j_2) - \quad \cdots \quad - F(j_{d-1} - j_d)$

$= 4d - 8 - 4(d-1) = -4$

If no vertex with 3 neighbors, $\mathcal{Q}$ is type A

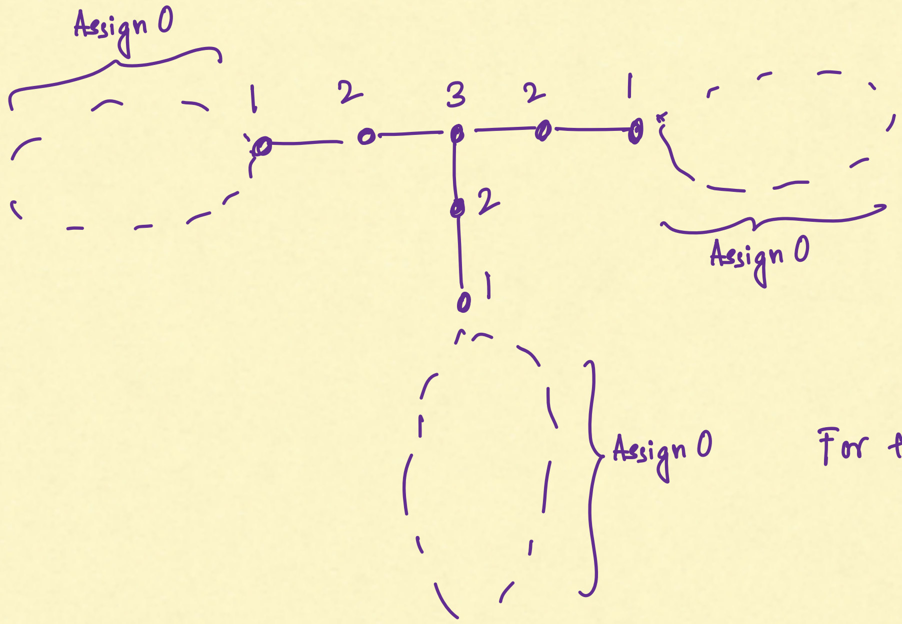
Otherwise :



WLOG $p \leq q \leq r$

Claim: $p < 2$

Pf. If not then all branches are at least 2 long.



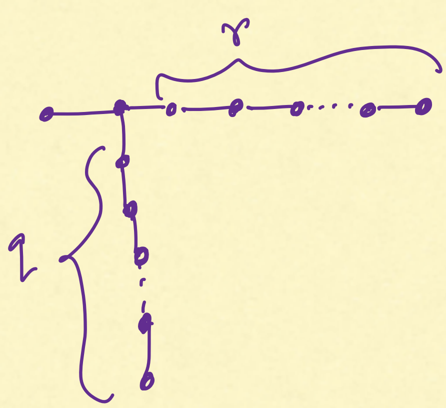
: Take α to be this assignment.

For this assignment $B(\alpha) = 0$.

So $p = 0$ or 1.

$p = 0$: type A

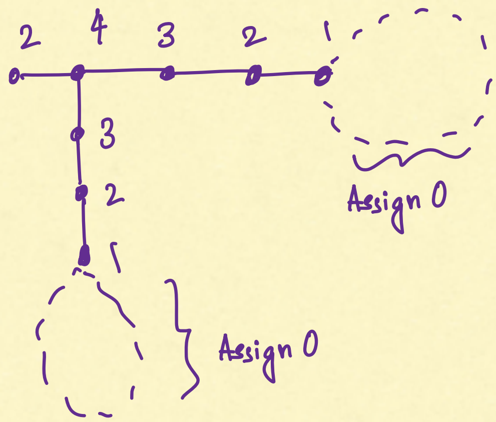
$p = 1$.



$1 \leq q \leq r$

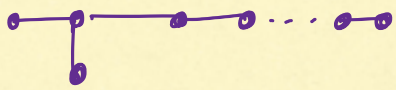
Claim: If $p=1$ then $q \le 2$

Pf.



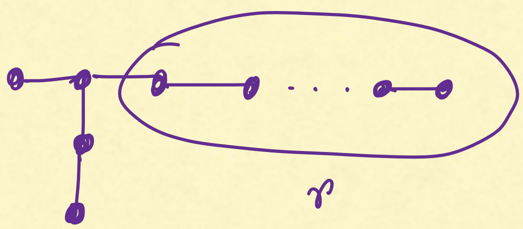
For this assignment of vectors, $B(\alpha)=0$

If $q=1$:



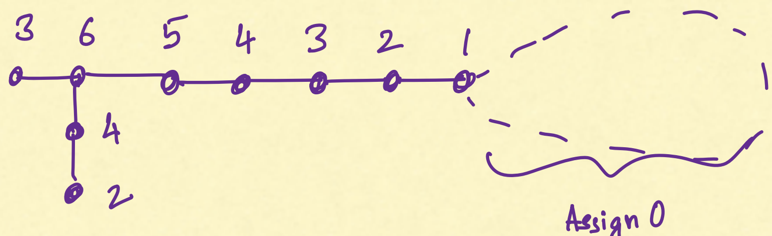
D_n

If $q=2$:



$2 \le r \le 4$

If $r \geq 5$:



This α gives $B(\alpha)=0$.

This gives E_6, E_7, E_8.

This proves ② ⇒ ③.

For ③ ⇒ ②, manually verifiable (not very insightful).

How to prove ① ⇒ ②? (Note: finitely many indecomposables for any dim vector is equiv to finitely many reps for any dim vector).

Say $Q=(V,\mathcal{E})$ is a rep of finite type. Fix a dim vector $\alpha \in \mathbb{N}_0^V$. $\alpha \neq (0,0,\dots,0)$.

So $Rep(Q, \alpha)$ is finite dim'l V.S.

Rep's of Q with dim vector α.

How it looks like? Given α, so $V_i = k^{\alpha(i)}$

and for each edge $e: i \to j$ we can freely choose a linear map $\varphi(e) \in Hom(V_i, V_j)$

So $X := Rep(Q,\alpha) \simeq \prod_{e\in\mathcal{E}} Hom(V_i, V_j)$

$\Rightarrow \dim Rep(Q,\alpha) = \sum_{e\in\mathcal{E}} \alpha(i).\alpha(j)$

Now there is a natural action of $G := \left\{\prod_{i\in V} GL_{\alpha(i)}(k)\right\} / k^* id$ on X

satisfying: ○ $g.V \simeq V \quad \forall g \in G, V \in X$

○ $V, W \in X$ s.t. $V \simeq W \Rightarrow \exists g \in G$ s.t. $gV \simeq W$.

So the collection of different non-isom reps with dim vector α are $GV_1, \dots, GV_n$ with $V_i \not\simeq V_j$ for $i \neq j$.

$$\dim G.V = \dim G - \dim G_V$$

$$\Rightarrow \dim G - \dim G.V = \dim G_V \geq 0$$

$$\Rightarrow \dim G \geq \dim G.V$$

Orbits are algebraic varieties, X is disjoint union of the finitely many distinct orbits, which are varieties.

dim of X is defined to be the max'l size of a chain of top subspaces of X : $X_1 \subseteq X_2 \subseteq \dots \subseteq X_n$, X_i irred.

But this also stands for dim of the orbits.

So some orbit $G \cdot x$ must have same dim as dim X.

$$\begin{aligned}
\therefore \quad \underset{\substack{\vee\! \mid \\ 0}}{\dim G_x} &= \dim G - \dim G \cdot x \\
&= \dim G - \dim X \\
&= \sum_{i \in V} \alpha(i)^2 - 1 - \sum_{e \in \mathcal{E}} \alpha(s(e))\, \alpha(t(e)) \\
&= B(\alpha) - 1
\end{aligned}$$

$\Rightarrow \; B(\alpha) \geqslant 1$