“Root and community inference on the latent growth process of a network”

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Want to find the source.

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- The generation looks something like this:
  - $\boldsymbol{T}_1 = \big([1], \{\}\big)$
  - Given $\boldsymbol{T}_{t-1}$, add a node labelled t and a random edge (t, w_t) to get $\boldsymbol{T}_t$ where w_t is chosen with probability $\dfrac{\beta \cdot D_{\boldsymbol{T}_{t-1}}(w_t) + \alpha}{2\beta(t-2) + \alpha(t-1)}$.

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- APA(1,0) gives the probability $\frac{1}{t-1}$. So a neighbor is chosen uniformly from $V(\boldsymbol{T}_{t-1})$.
- APA(0,1) gives the probability $\frac{D_{T_{t-1}}(w_t)}{2(t-2)}$. So a neighbor is chosen with probability proportional to its degree.

PAPER(α, β, θ)

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We say that a random graph G_n is distributed accordion to $\mathrm{PAPER}(\alpha, \beta, \theta)$ if $\boldsymbol{G}_n = \boldsymbol{T}_n + \boldsymbol{R}_n$ if $\boldsymbol{T}_n \sim \mathrm{APA}(\alpha, \beta)$ and $\boldsymbol{R}_n \sim \mathrm{Erdös - Rényi}(\theta)$.

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Will drop subscript

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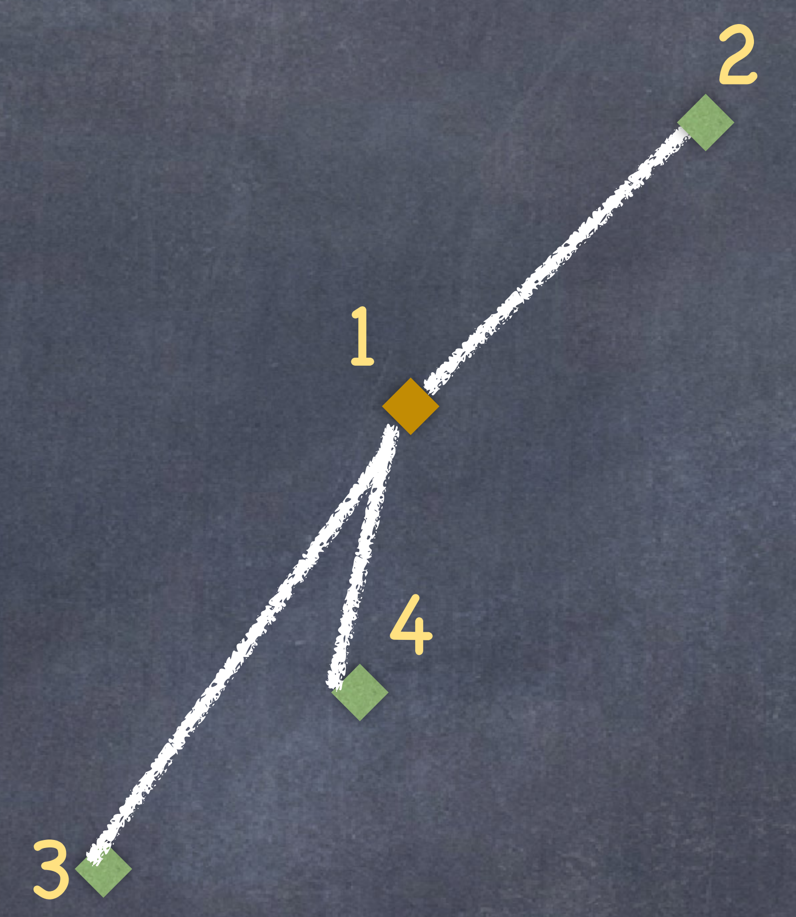

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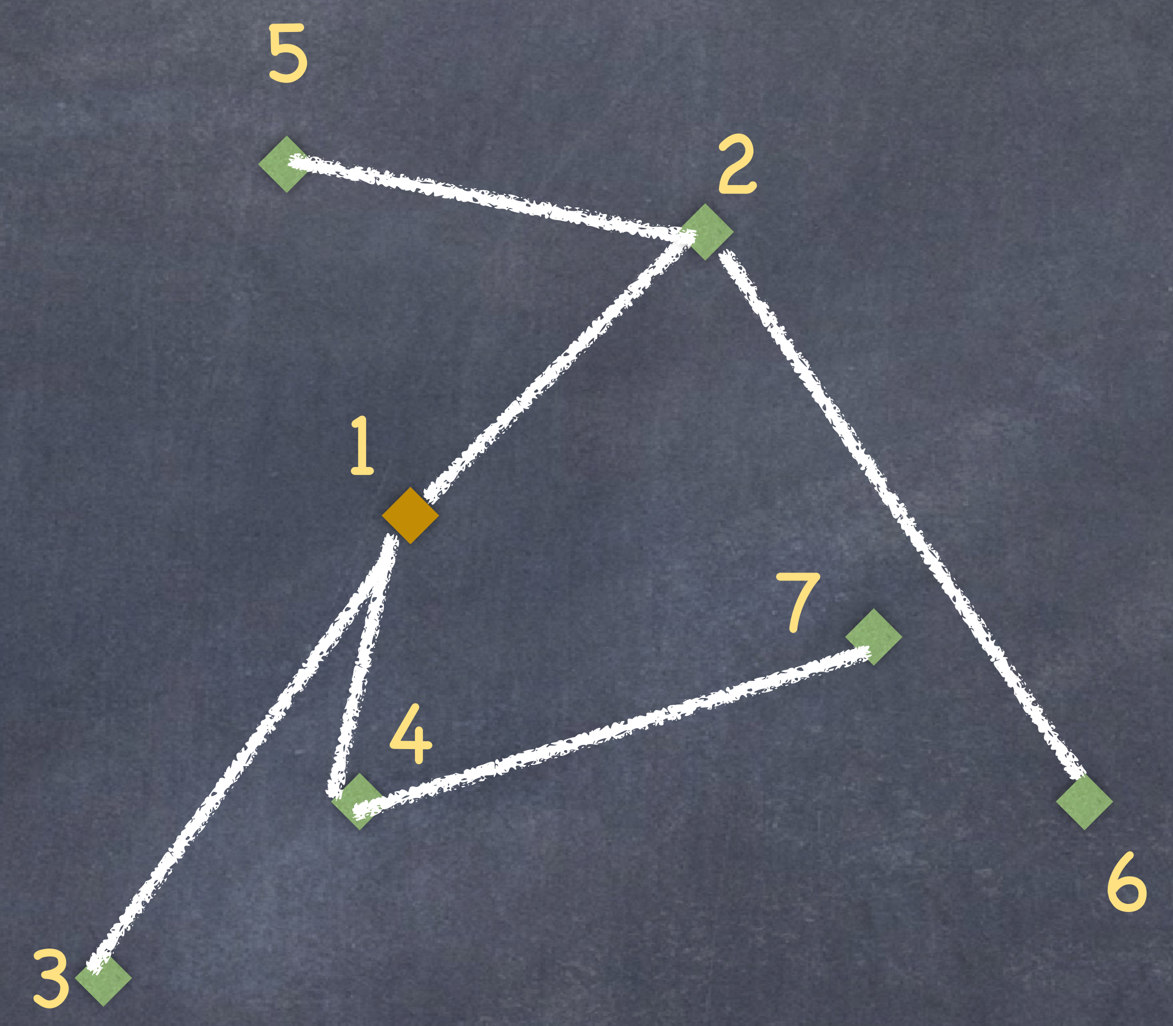

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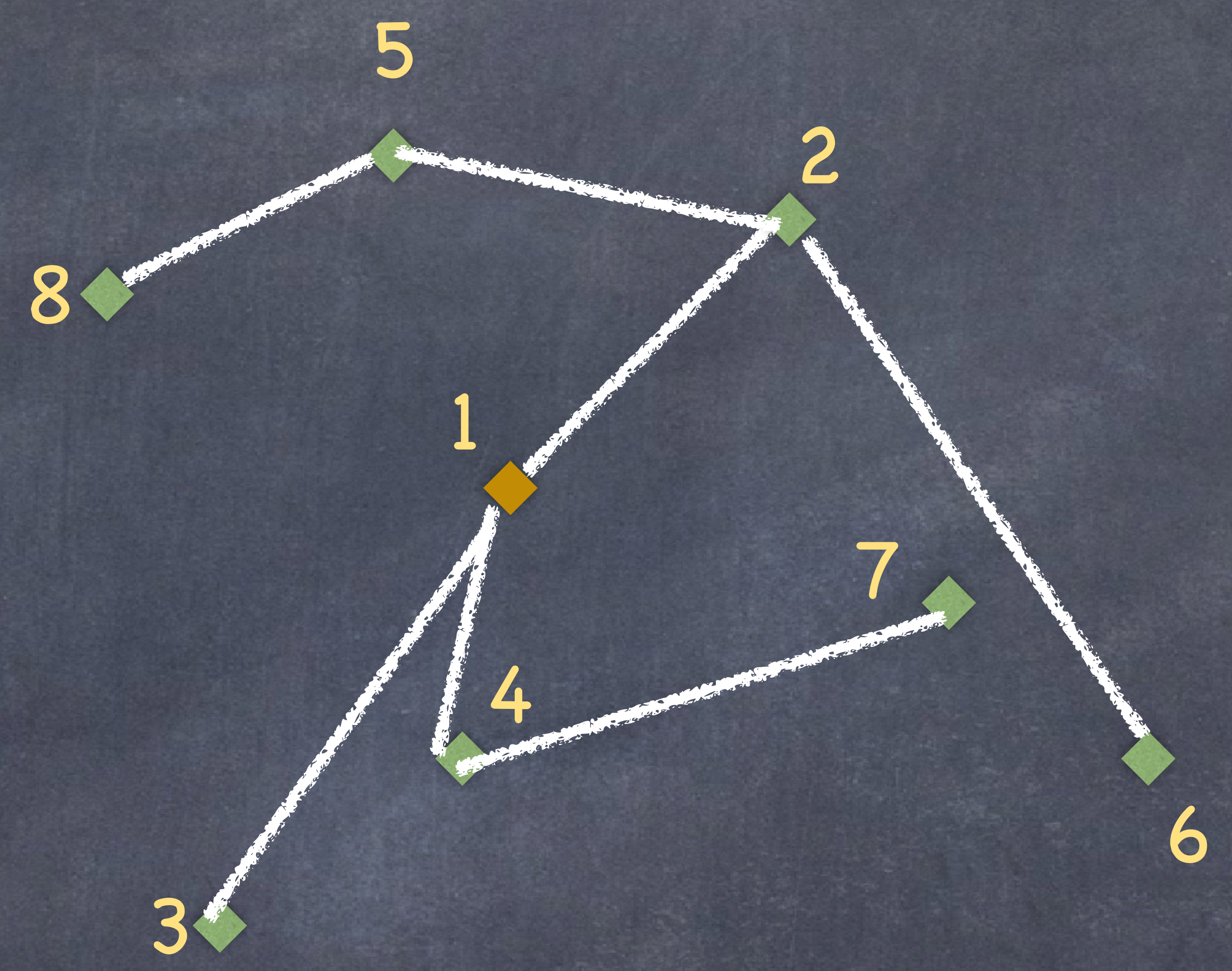

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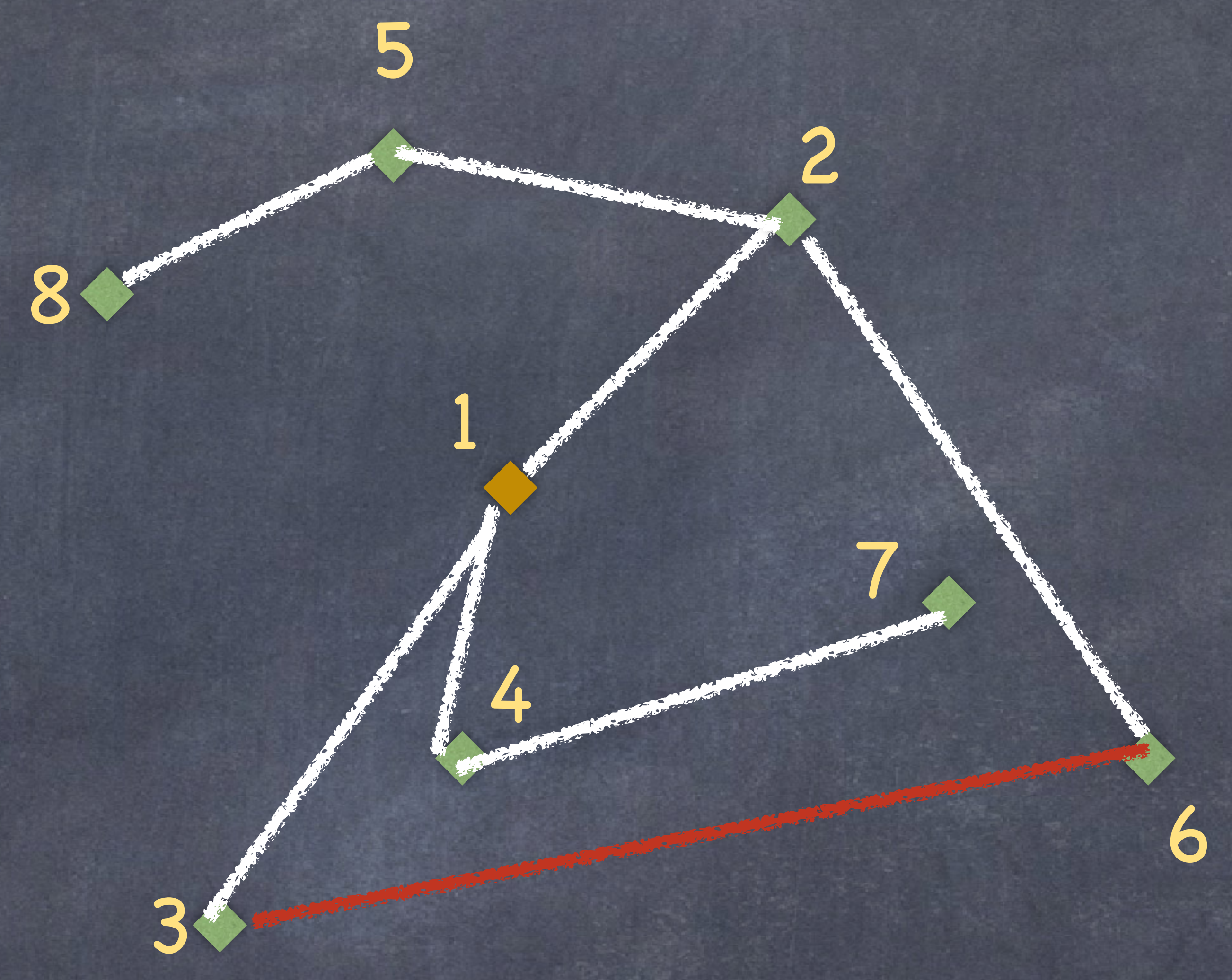

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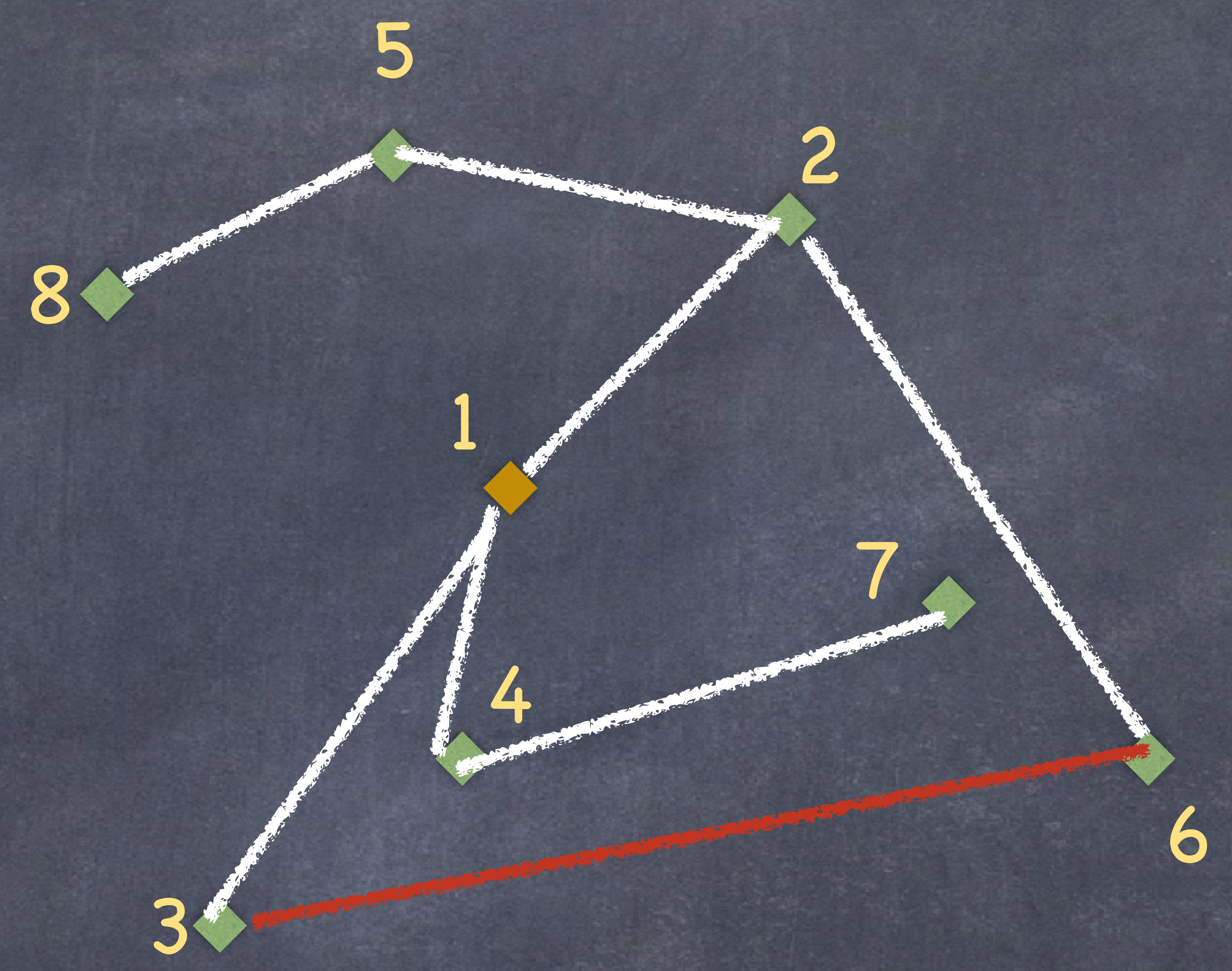

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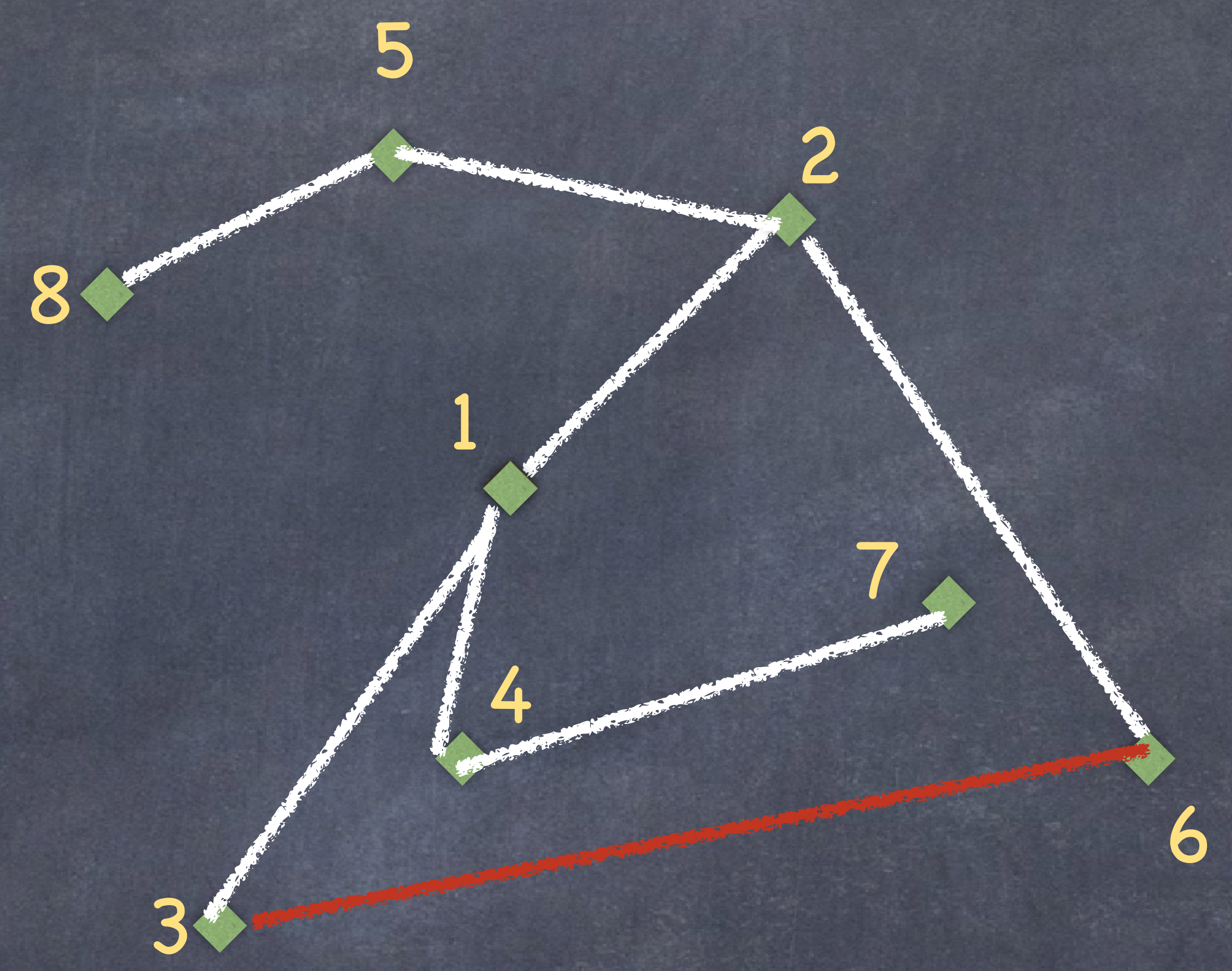

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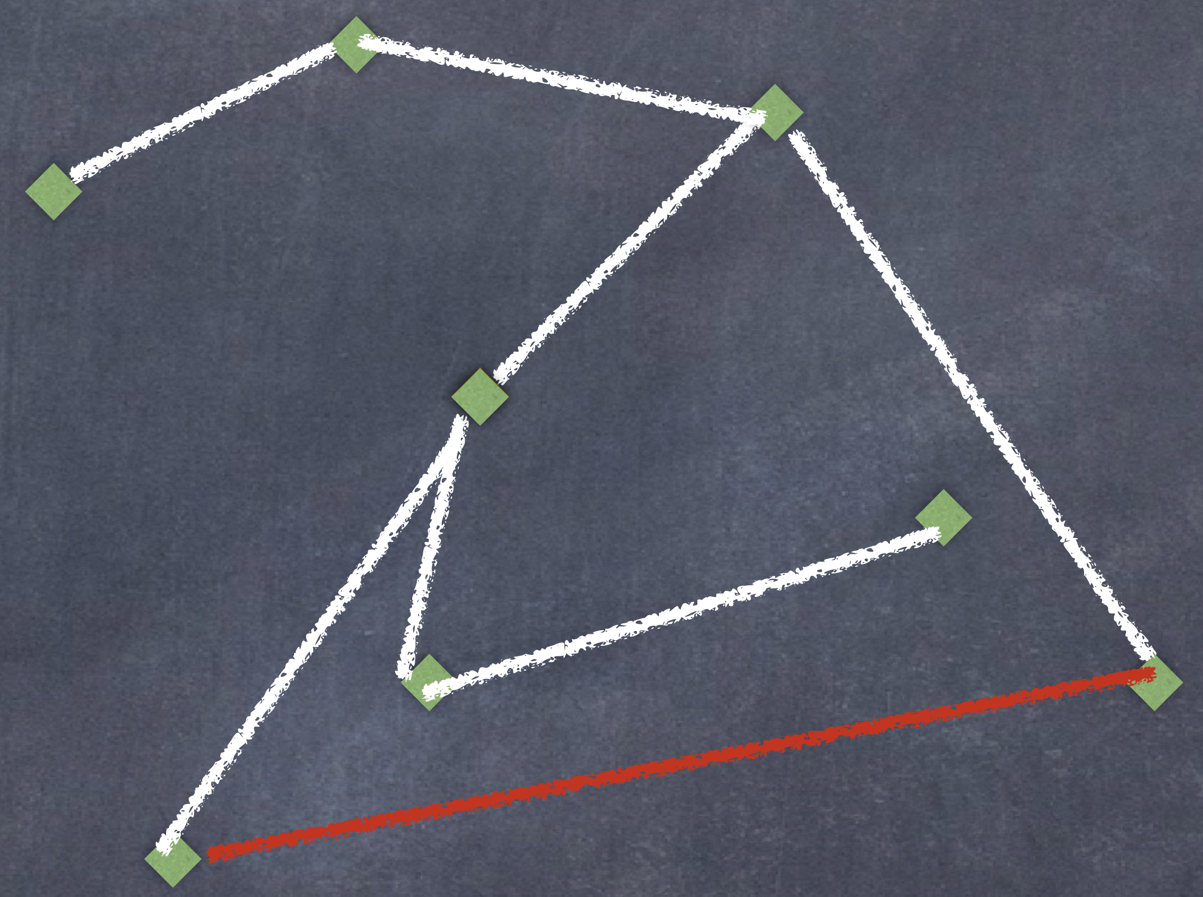

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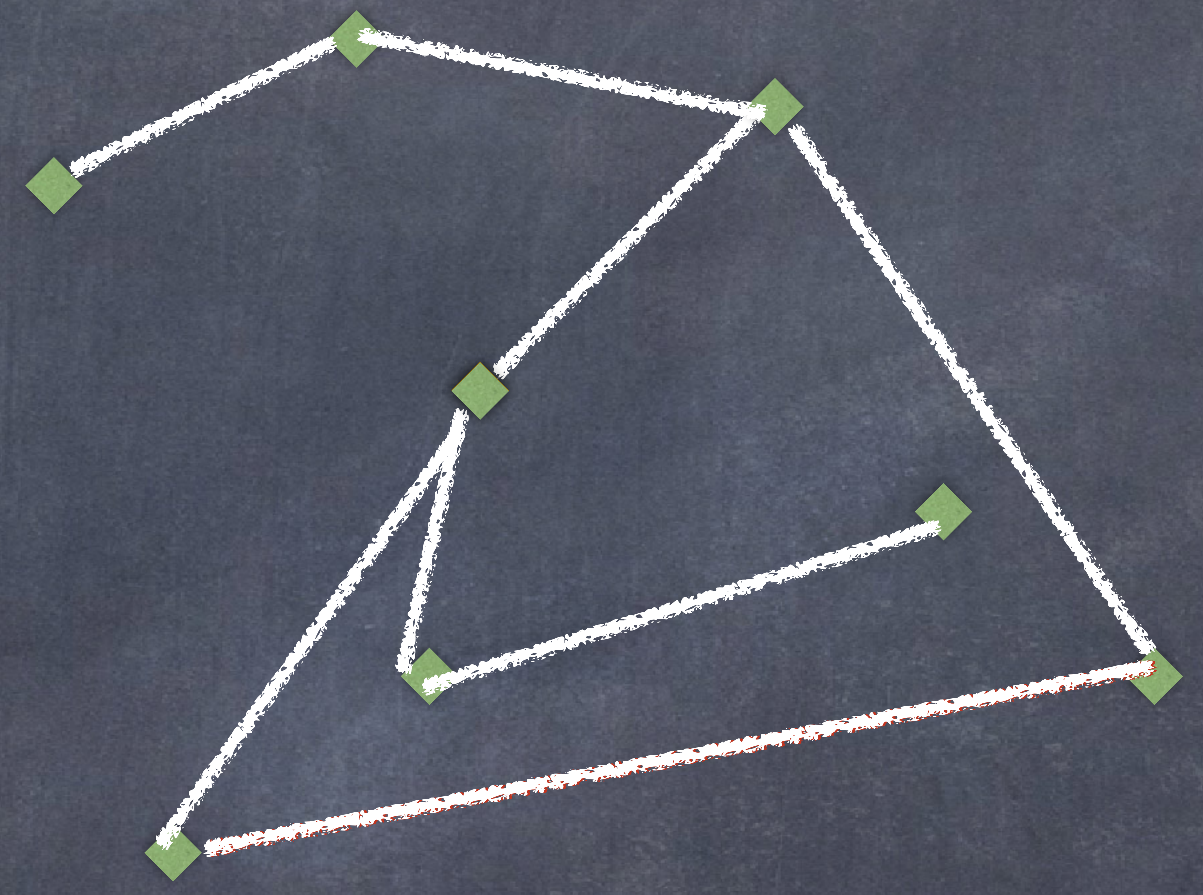

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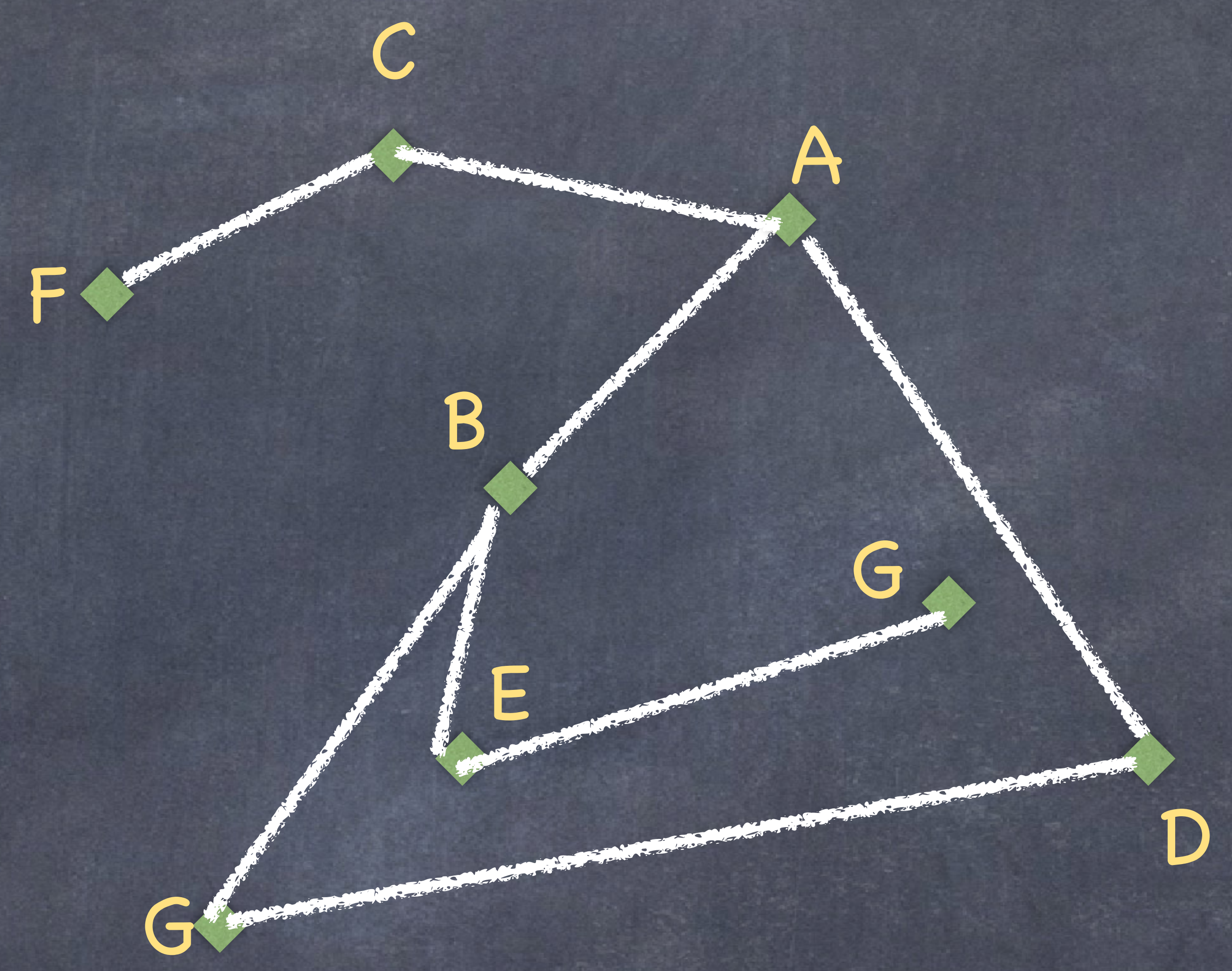

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- **More concretely:**
  Give me a set of vertices $C(\boldsymbol{G}^*) \subseteq V(\boldsymbol{G}^*)$ such that $\mathbb{P}(\blacklozenge \in C(\boldsymbol{G}^*)) \geq 95\,\%$.

- So our goal now:

  Given $\epsilon \in (0,1)$, find $C_\epsilon \subseteq V = \{A, B, \cdots\}$ such that $\mathbb{P}(\blacklozenge \in C_\epsilon(\boldsymbol{G}^*)) \geq 1 - \epsilon$.

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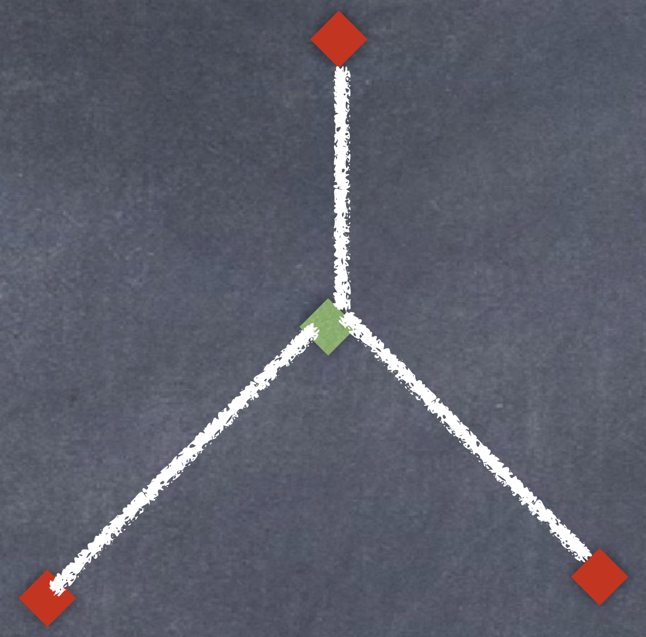

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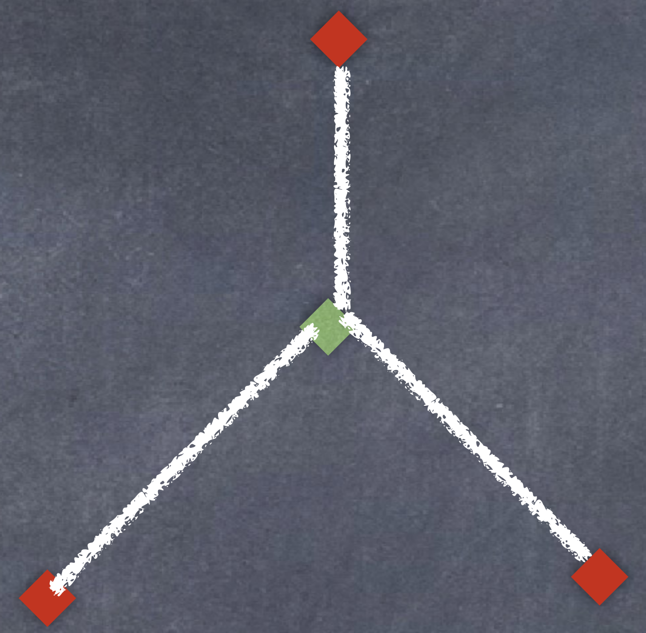

C_ϵ either contains all ◆ or contains none of them.

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Will compute this

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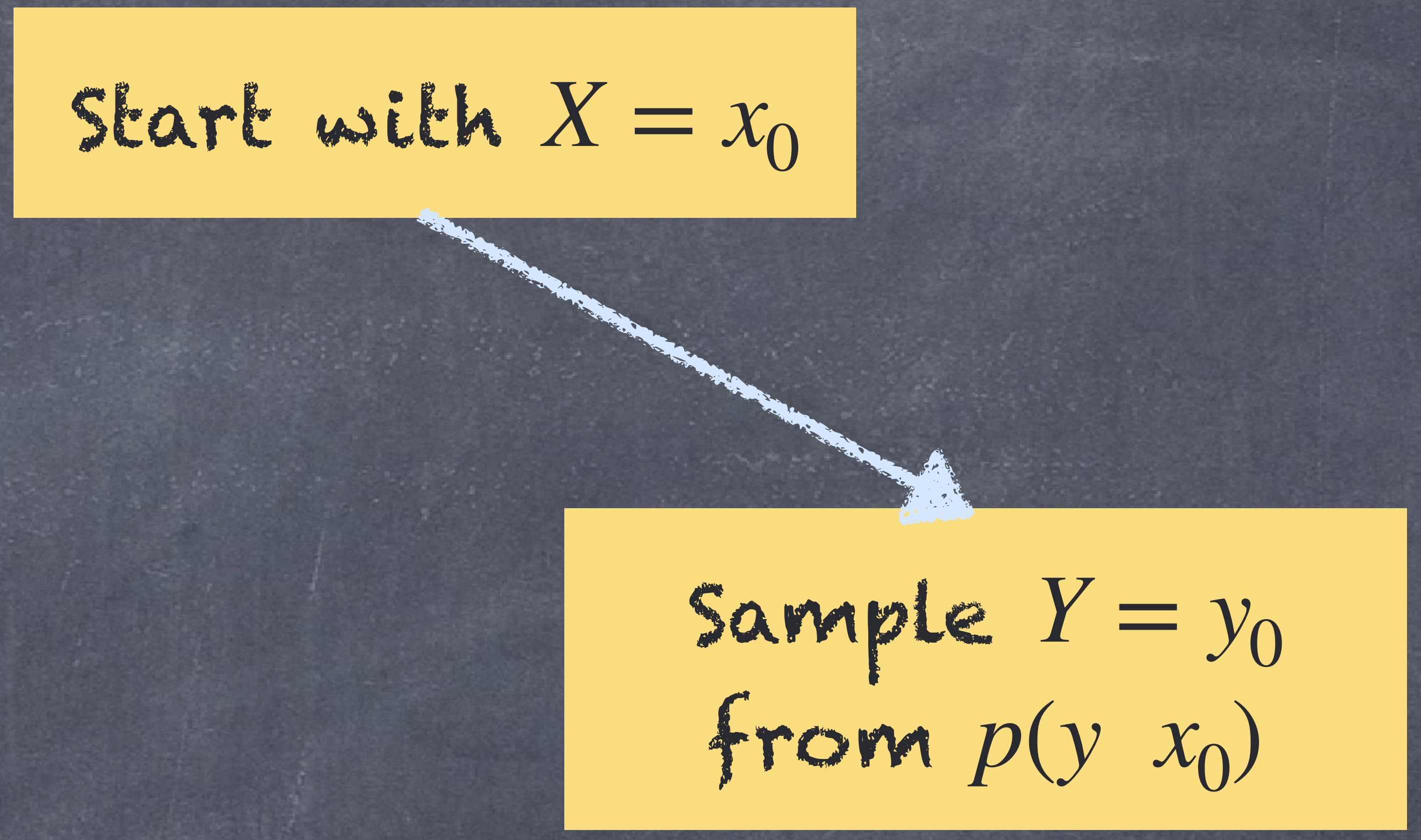
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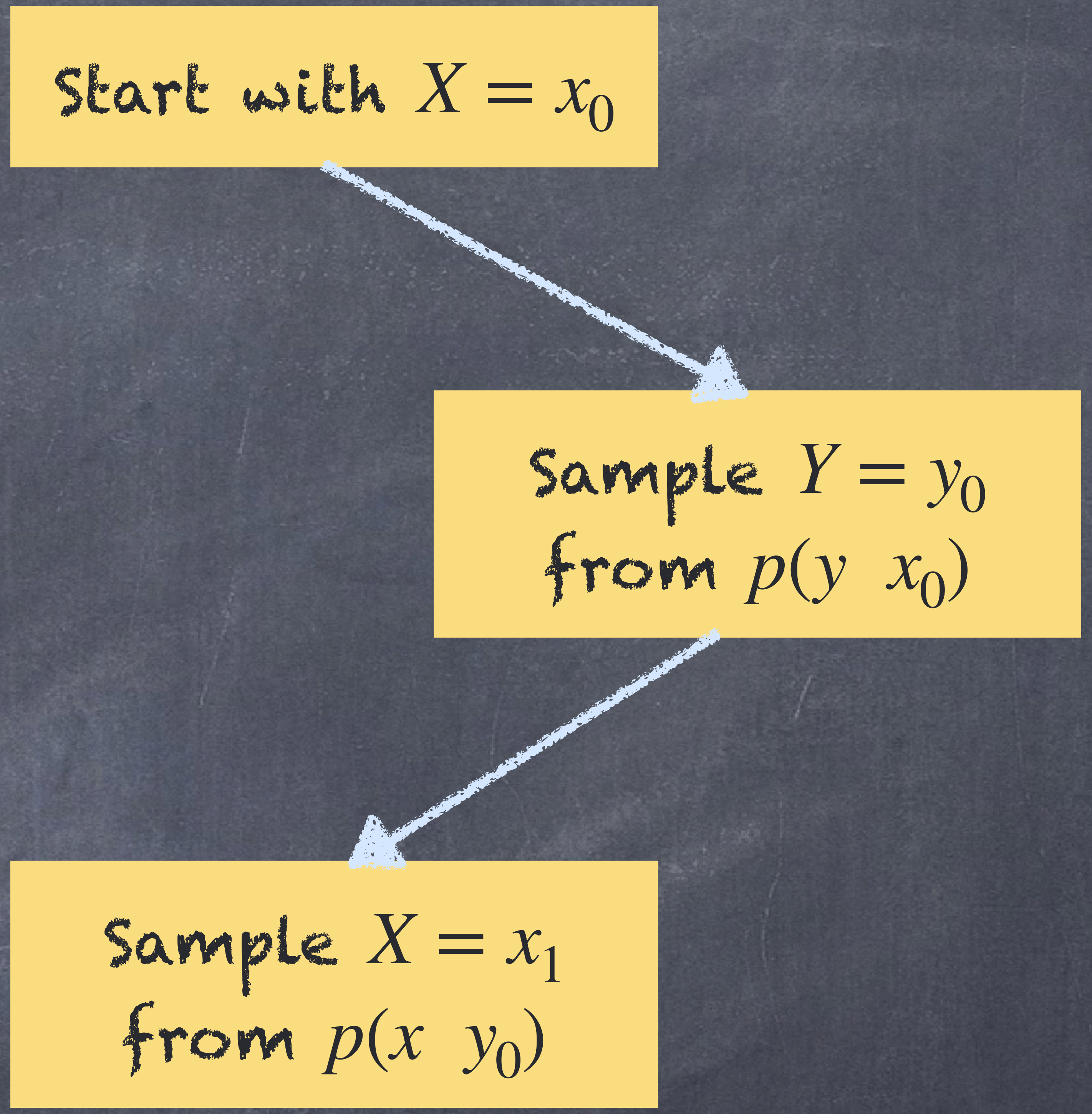

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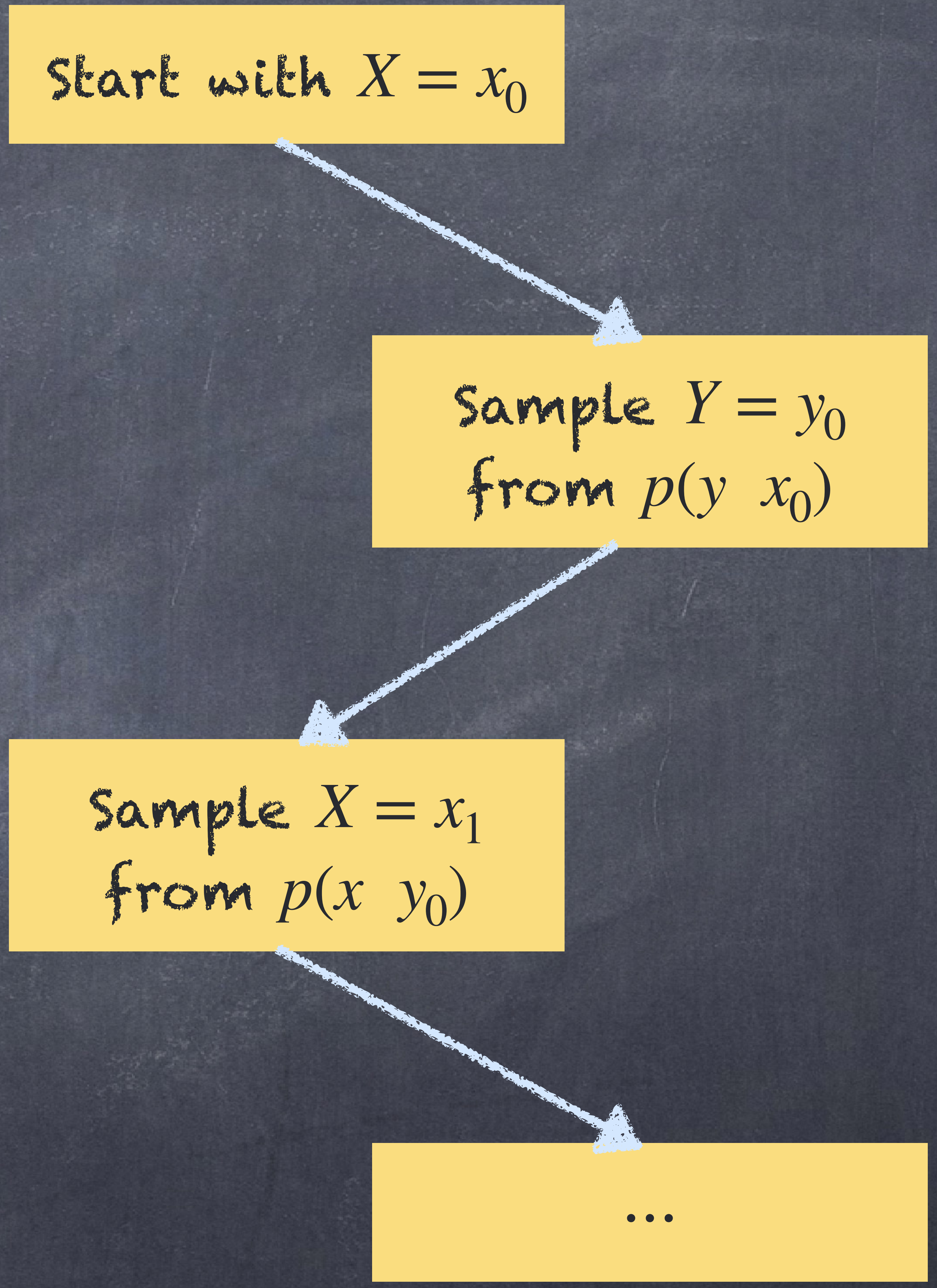

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- Remember that $X = \tilde{T}, Y = \Pi$.
- So we alternate between two stages:
  - Fix t and generate π from distribution $\mathbb{P}\left(\Pi = \pi \;\; \tilde{T} = t, \tilde{G} = g\right)$.
  - Fix π and generate t from the distribution $\mathbb{P}\left(\tilde{T} = t \;\; \Pi = \pi, \tilde{G} = g\right)$.

An example (taken from the manuscript)

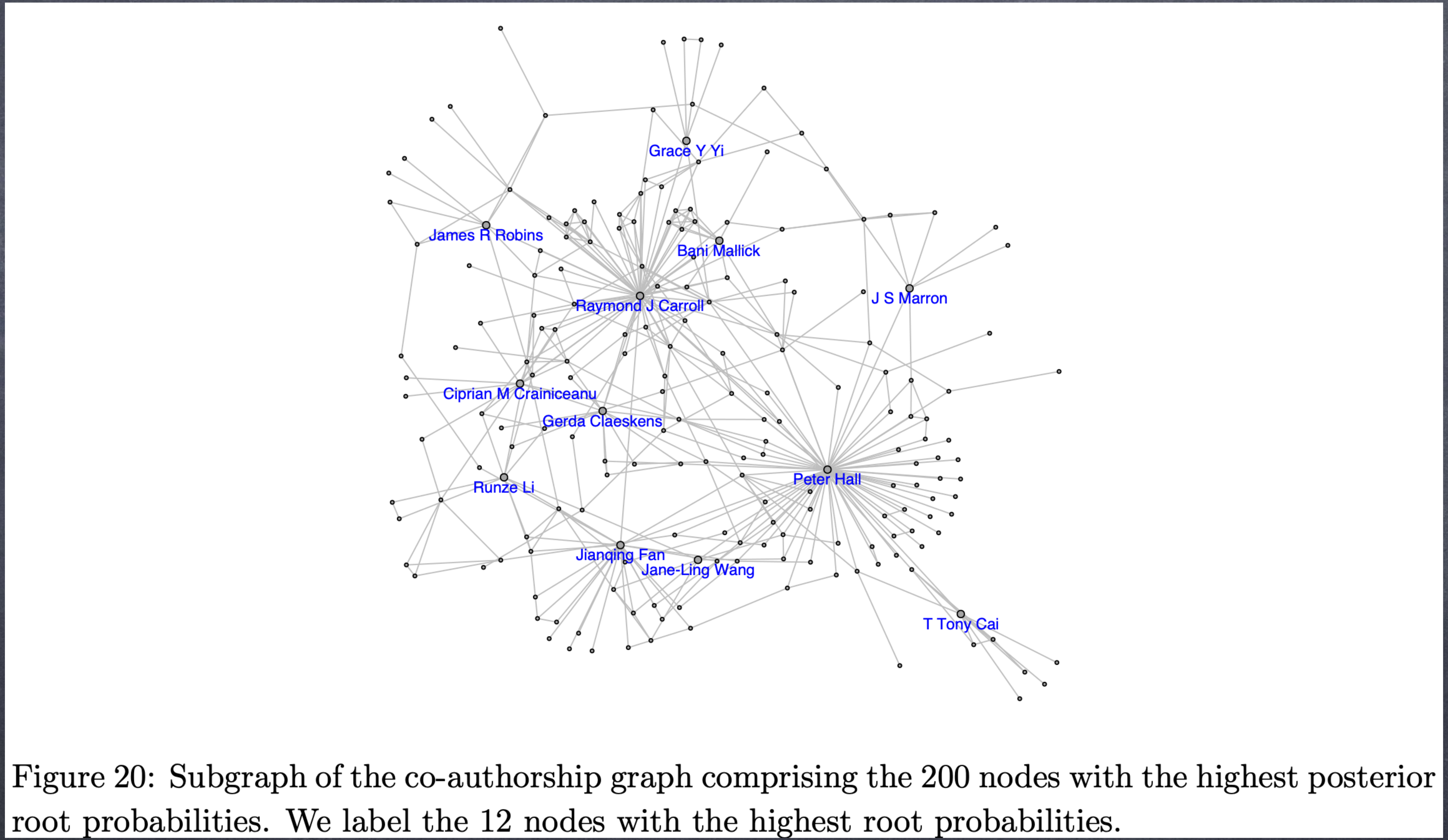

Figure 20: Subgraph of the co-authorship graph comprising the 200 nodes with the highest posterior root probabilities. We label the 12 nodes with the highest root probabilities.

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