Complexity of optimization

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October 21, 2023

Maximize $f = x + y + z$ subject to $g = x^4 + y^4 + 3z^4 - z - 1 = 0$

$\mathcal{L} = (x + y + z) + \lambda(x^4 + y^4 + 3z^4 - z - 1)$.

$$\partial_x \mathcal{L} = 1 + 4\lambda x^3$$
$$\partial_y \mathcal{L} = 1 + 4\lambda y^3$$
$$\partial_z \mathcal{L} = 1 + \lambda(12z^3 - 1)$$
$$\partial_\lambda \mathcal{L} = g$$

Trying to define an ideal in `SageMath` given by the above generators and finding a Gröbner basis tells us that we need to solve an equation of degree 36.
If we add another **generic** linear constraint, this degree is now 12.
Another **generic** linear constraint makes the degree 4.
Adding another **generic** equation means that there's no solution, which gives degree 0.

Define these numbers to be the algebraic degrees: $d_1 = 36, d_2 = 12, d_3 = 4$.

An example

Maximize $f = x + y + z$ subject to $g = x^4 + y^4 + 3z^4 - z - 1 = 0$

$\mathcal{L} = (x + y + z) + \lambda(x^4 + y^4 + 3z^4 - z - 1)$.

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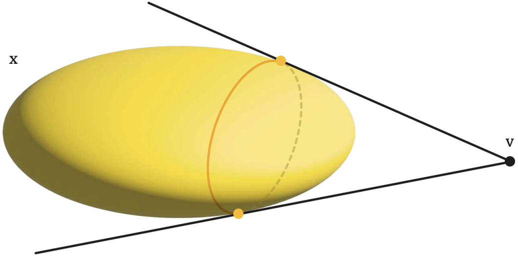
Polar Variety

Imagine a compact ellipsoid X and a point $V = \boldsymbol{v}$. Imagine that your eyes are at $\boldsymbol{v}$. What do you see? (Picture taken from the book Metric Algebraic Geometry by Breiding, Kohn, Sturmfels.)



Suppose $X \subseteq \mathbb{P}^3$ is given by a homogeneous polynomial f of degree d and $\boldsymbol{v} = (v_0 : v_1 : v_2 : v_3)$ is the point where your eyes are. It is a curve, name it $P(X, \boldsymbol{v})$, is determined by f and $\partial_{\boldsymbol{v}} f$.

Theorem (Bezout)

Let $f_1, \cdots, f_k$ be general polynomials in n variables of degree $d_1, \cdots, d_k$ respectively. For $I = \langle f_1, \cdots, f_k \rangle$ we have $\dim I = n - k$ and $\deg I = d_1 \cdots d_k$.

So this $P(X, V)$ *typically* has degree $d(d-1)$.

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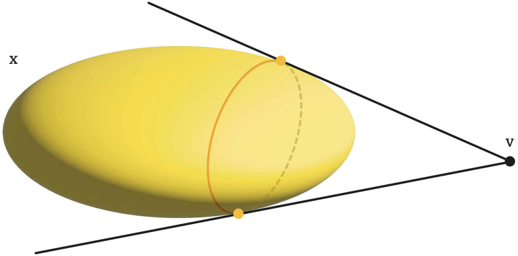

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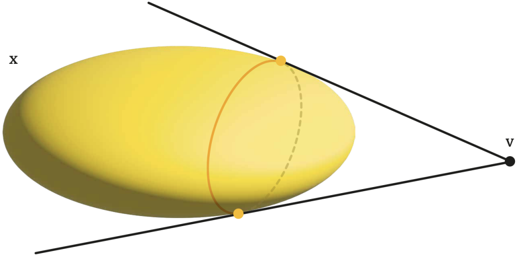

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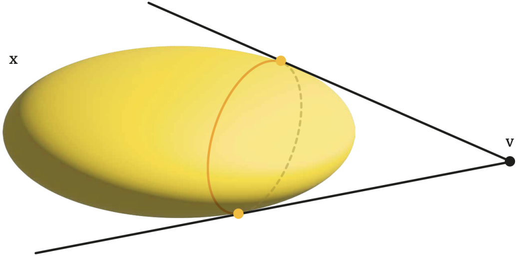

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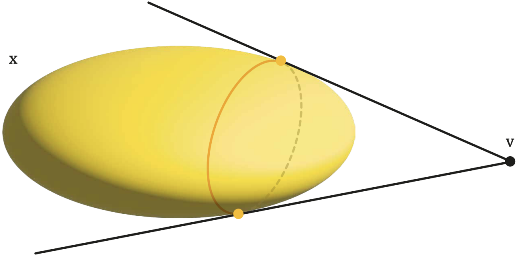

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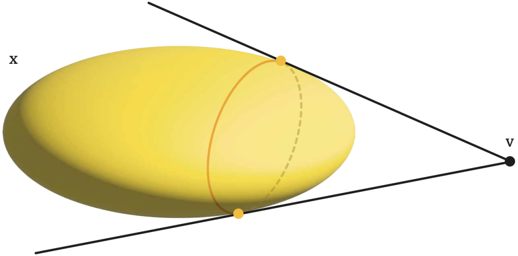

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Polar degrees

Definition (Polar Variety)

The polar variety of a variety $X \subseteq \mathbb{P}^n$ with respect to a projective subspace $V \subseteq \mathbb{P}^n$ is

$$P(X,V) = \overline{\{\boldsymbol{p} \in \operatorname{Reg}(X) \setminus V : V + \boldsymbol{p} \text{ intersects } X \text{ at } \boldsymbol{p} \text{ non-transversally}\}}.$$

Let $i \in \{0, 1, \cdots, \dim X\}$. If V is generic with $\dim V = \operatorname{codim}(X) - 2 + i$, then the degree of $P(X,V)$ is independent of V:

$$\mu_i(X) = \deg P(X,V).$$

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For a general optimization problem

Given a compact smooth algebraic variety $\mathcal{M}$ in $\mathbb{R}^m$, we consider a linear functional ℓ and an affine-linear space L of codimension r in $\mathbb{R}^m$. It is assumed that the pair (ℓ, L) is in general position[*] relative to $\mathcal{M}$. Our aim is to study the following optimization problem:

$$\text{maximize } \ell \text{ over } L \cap \mathcal{M}.$$

[†]

Theorem

The algebraic degree of the above problem is $\mu_r(\mathcal{M})$.

[*] this assumption is very important

[†] Türkü Özlüm Çelik, Asgar Jamneshan, Guido Montúfar, Bernd Sturmfels, and Lorenzo Venturello. "Wasserstein distance to independence models". In: *Journal of symbolic computation* 104 (2021), pp. 855–873.

Wasserstein distance

Let $d : [n] \times [n] \to \mathbb{R}_{\geq 0}$ be a metric on $[n]$. Consider the polytope

$$B_d = \operatorname{conv}\left\{ \frac{1}{d_{ij}}(\boldsymbol{e}_i - \boldsymbol{e}_j) : i \neq j \right\} \subseteq \underbrace{\{\mathbf{1}^\top \boldsymbol{x} = 0\}}_{H_{n-1}} \subseteq \mathbb{R}^n.$$

B_d is compact, convex, origin symmetric and has nonzero relative interior on H_{n-1}. Thus induces a norm on H_{n-1}, and thus a metric on any affine hyperplane perpendicular to $\mathbf{1}$. In particular, given any two probability vectors $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Delta_{n-1}$, we define their Wasserstein distance based on d to be

$$W_d(\boldsymbol{\mu}, \boldsymbol{\nu}) = \inf \{ t > 0 : \boldsymbol{\nu} \in \boldsymbol{\mu} + tB_d \}.$$

We will consider distance of a point $\boldsymbol{\mu}$ from a *statistical* model $\mathcal{M}$, namely,

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Consider two random variables $X_1, X_2 \sim$ Bernouli(p) and $N = X_1 + X_2$. So N =number of heads on tossing a coin twice independently which has probability p of getting a head. Associate with it the curve given by $(\mathbb{P}[N=0], \mathbb{P}[N=1], \mathbb{P}[N=2]) = \left(p^2, 2p(1-p), (1-p)^2\right)$ for $p \in [0,1]$.

To get to one more level of abstraction, we consider models of the form $\mathcal{M} = \Delta_{n-1} \cap X$ for some affine variety X.

As long as $\boldsymbol{\mu}$ and $\mathcal{M}$ are generic with respect to B_d, there will be a unique intersection point, and it will be in the relative interior of one of the faces F of B_d. We let $\mathcal{L}_F$ be the linear subspace generated by the vertices of the face F of B and let ℓ_F be any linear functional that attains its maximum over B at F. Then the optimal solution to

$$\text{minimize } \ell_F(\boldsymbol{\nu}) \text{ subject to } \boldsymbol{\nu} \in (\boldsymbol{\mu} + \mathcal{L}_F) \cap \mathcal{M}$$

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Polar degree again

We need to consider this optimization problem for every face F of B_d. Also $\mathcal{M} = \Delta_{n-1} \cap X$. Recall the numbers $36, 12, 4, 0$ from the earlier example on Lagrange multipliers. We study these Wasserstein degrees $w(X, F)$.

By the earlier theorem we stated, these numbers are always upper bounded by the polar degree $\mu_i(X)$ where $i = dim(X) - codim(F) + 1$.

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Result for rational normal scroll

Consider the rational normal scroll $S = S(n_1, \cdots, n_k)$ in $\mathbb{P}^{n-1}$ whose ideal is generated by the 2×2 minors of $M = \begin{bmatrix} M_{n_1} & M_{n_2} & \cdots & M_{n_k} \end{bmatrix}$ where $M_{n_j} = \begin{bmatrix} x_{j,0} & \cdots & x_{j,n_j-1} \\ x_{j,1} & \cdots & x_{j,n_j} \end{bmatrix}$. One can check that S has dimension k and degree $N = \sum_{i=1}^{k} n_i$.

If $k = 2$, we recover the Hirzebruch surface $S(a, b)$.

Theorem (DHMN'24)

Let $S = S(n_1, \cdots, n_k) \subseteq \mathbb{P}^{n-1}$ be a rational normal scroll and let $N = \sum_{i=1}^{k} n_i$. The polar degrees of S are

$$\mu_j = \begin{cases} N & \text{if } j = k\mathbf{1} \\ 2(N-1) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}.$$

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Thank You :)

Definition (Conormal variety)

The *conormal variery* $N_X \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is the Zariski closure of of the collection of all pairs $(\boldsymbol{x}, \boldsymbol{h}) \in \mathbb{P}^n \times \mathbb{P}^n$ such that $\boldsymbol{x}$ is a non-singular point in X and $\boldsymbol{h}$ represents a hyperplane tangent to X at $\boldsymbol{x}$.

Now $H^*(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[s,t]/\left\langle s^{n+1}, t^{n+1} \right\rangle$. The class of the conormal variety N_X in this cohomology ring is a binary form of degree $n+1 = \text{codim}(N_X)$ whose coefficients are nonnegative integers:

$$[N_X] = \sum_{i=1}^{n} \delta_i(X) s^{n+1-i} t^i$$

.

Theorem

$\delta_i(X) = \mu_i(X)$.