QUANTUM COMPUTATION

Lecture 3

Nilava Metya
nilavam@cmi.ac.in

26 July 2020

1 Quantum entanglement (examples)

Exercise 1
Show that a (natural) inner product on $H \otimes H$ is $\langle \alpha\beta|\gamma\delta\rangle = \langle \alpha|\gamma\rangle \langle \beta|\delta\rangle$.

Exercise 2
Let $|\alpha\rangle, |\beta\rangle$ be an orthonormal basis of $H = \mathbb{C}^2$ and consider the observable $X = 0\,|\alpha\rangle\langle\alpha| + 1\,|\beta\rangle\langle\beta|$. Measuring $X \otimes \mathbf{1}$ with respect to the Bell state $|\Phi^+\rangle$ forces the second qubit to collapse to the conjugate of same state as the first qubit. Measuring with respect to $|\Psi^-\rangle$ does the opposite. Try for $|\Phi^-\rangle$ and $|\Psi^+\rangle$.

Solution. Let $|\alpha\rangle = a\,|0\rangle + b\,|1\rangle$ for some $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$. So $|\beta\rangle = d\,|0\rangle - c\,|1\rangle$ such that $(c, d) = e^{i\theta}(a^*, b^*)$ for some θ. We have $X = 0\,|\alpha\rangle\langle\alpha| + 1\,|\beta\rangle\langle\beta| = 0E_\alpha + 1E_\beta$ When we measure $X \otimes \mathbf{1}$ in the state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we have $\text{Prob}_{|\Phi^+\rangle}(X \otimes \mathbf{1} = 0) = \langle \Phi^+|(E_\alpha \otimes \mathbf{1})\Phi^+\rangle$. Some small results: $\langle\alpha|0\rangle = a^*, \langle\alpha|1\rangle = b^*, \langle 0|\alpha\rangle = a, \langle 1|\alpha\rangle = b$. Firstly note that

$$\begin{aligned}(E_\alpha \otimes \mathbf{1})\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}\left[(a^*\,|\alpha\rangle) \otimes |0\rangle + (b^*\,|\alpha\rangle) \otimes |1\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[|\alpha\rangle \otimes (a^*\,|0\rangle) + |\alpha\rangle \otimes (b^*\,|1\rangle)\right] \\ &= \frac{1}{\sqrt{2}}\,|\alpha\rangle \otimes (a^*\,|0\rangle + b^*\,|1\rangle) = \frac{1}{\sqrt{2}}\,|\alpha\rangle \otimes |\overline{\alpha}\rangle\end{aligned}$$

And this gives us $\text{Prob}_{|\Phi^+\rangle}(X \otimes \mathbf{1} = 0) = \left\langle \Phi^+ \middle| (E_\alpha \otimes \mathbf{1})\Phi^+ \right\rangle = \dfrac{1}{2}$. And so $\left|\tilde{\Phi}^+\right\rangle = |\alpha\rangle \otimes |\overline{\alpha}\rangle$.

Exercise 3
Let $|\alpha\rangle, |\beta\rangle$ be an orthonormal basis of $H = \mathbb{C}^2$. Then the Bell state $|\Phi^+\rangle$ can be written as $\left|\Phi^+\right\rangle = \dfrac{1}{\sqrt{2}}\left(|\alpha\overline{\alpha}\rangle + \left|\beta\overline{\beta}\right\rangle\right)$.
Try a similar thing for the other Bell states.

2 The commutator and anti-commutator

Definition 2.1. The *commutator* between two operators A, B is $[A, B] := AB - BA$.

Definition 2.2. The *anti-commutator* between two operators A, B is $\{A, B\} := AB + BA$.

EXERCISE 4
Check the following for operators A, B, C and scalars a, b:

1. $[aA + bB, C] = a[A, C] + b[B, C], [C, aA + bB] = a[C, A] + b[C, B]$
2. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$
3. $[A, B] = -[B, A]$
4. $[A, B]^* = [B^*, A^*]$
5. $[A, BC] = [A, B]C + B[A, C], [AB, C] = A[B, C] + [A, C]B$
6. $AB = \dfrac{[A, B] + \{A, B\}}{2}$
7. If A, B are self-adjoint, so is $i[A, B]$.

Theorem 2.3 (Simultaneous diagonalization theorem)**.** *Suppose A, B are hermitian operators. Then $[A, B] = 0$ iff they can be simultaneously diagonalized with respect to some (common) orthonormal basis.*

Remark 2.4. The Lie brackets used here is the quantum mechanics equivalent of the Poisson brackets used in classical mechanics. It might be a good thought exercise to figure out how the simultaneous diagonalization theorem has a physical meaning!

3 Heisenberg's Uncertainty Principle

Definition 3.1. The *expected value* or *expectation* of an observable X with respect to a state $|\psi\rangle$ is $\langle X \rangle_\psi := \langle \psi | X | \psi \rangle$. Due to abuse of notation, we just write $\langle X \rangle$.

EXERCISE 5
Show that the above definition of expectation of a quantum observable is as good as the definition of expectation of a classical random variable, that is, show that if $X = \sum_{\lambda \in \sigma(X)} \lambda E_\lambda$ is the spectral decomposition (all E_λ's are rank one projections) then $\langle \psi | X | \psi \rangle = \sum_{\lambda \in \sigma(X)} \lambda \cdot \text{Prob}_\psi(X = \lambda)$.

Solution.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{\lambda \in \sigma(X)} \lambda \cdot \text{Prob}_\psi(X = \lambda) \\
&= \sum_{\lambda \in \sigma(X)} \lambda \langle \psi | E_\lambda | \psi \rangle \\
&= \sum_{\lambda \in \sigma(X)} \langle \psi | \lambda E_\lambda | \psi \rangle \\
&= \left\langle \psi \middle| \sum_{\lambda \in \sigma(X)} \lambda E_\lambda \middle| \psi \right\rangle \\
&= \langle \psi | X | \psi \rangle = \langle X \rangle
\end{aligned}$$

Definition 3.2. The standard deviation of an obvservable X with respect to the state $|\psi\rangle$ is defined as $\Delta_\psi(X) := \sqrt{\langle X^2\rangle_\psi - \langle X\rangle_\psi^2}$. Again due to abuse of notation, we simply write $\Delta(X)$.

Exercise 6
The above is a good definition of standard deviation, that is, verify that $(\Delta(X))^2 = \langle (X - \langle X\rangle)^2\rangle$.

Solution.

$$\begin{aligned}\langle (X - \langle X\rangle)^2\rangle &= \langle X^2 - 2\langle X\rangle X + \langle X\rangle^2\rangle \\ &= \left\langle \psi \middle| X^2 - 2\langle X\rangle X + \langle X\rangle^2 \middle| \psi\right\rangle \\ &= \langle \psi|X^2|\psi\rangle - 2\langle X\rangle \langle\psi|X|\psi\rangle + \langle X\rangle^2 \langle\psi|\psi\rangle \\ &= \langle X^2\rangle - 2\langle X\rangle^2 + \langle X\rangle^2 \\ &= \langle X^2\rangle - \langle X\rangle^2 = (\Delta(X))^2\end{aligned}$$

Theorem 3.3 (Heisenberg's inequality)**.** *If A, B are observables in a quantum system under the vector state $|\rho\rangle$, then*

$$\Delta(A)\cdot\Delta(B) \geq \frac{1}{2}\langle [A, B]\rangle$$

Proof. Let X, Y be any observables in a quantum system with state $|\rho\rangle$.
Say $\langle\rho|XY|\rho\rangle = a + ib$ for some $a, b \in \mathbb{R}$.
Then we have

$$\begin{aligned}\langle\rho|[X,Y]|\rho\rangle &= \langle\rho|XY|\rho\rangle - \langle\rho|YX|\rho\rangle = 2ib \\ \langle\rho|\{X,Y\}|\rho\rangle &= \langle\rho|XY|\rho\rangle + \langle\rho|YX|\rho\rangle = 2a\end{aligned}$$

And thus, $|\langle\rho|[X,Y]|\rho\rangle|^2 + |\langle\rho|\{X,Y\}|\rho\rangle|^2 = 4|\langle\rho|XY|\rho\rangle|^2$ Applying the Cauchy Schwarz inequality and have,

$$\begin{aligned} |\langle\rho|XY|\rho\rangle|^2 = |\langle X\rho|Y\rho\rangle|^2 &\overset{\text{CS}}{\leq} \langle X\rho|X\rho\rangle\langle Y\rho|Y\rho\rangle = \langle\rho|X^2|\rho\rangle\langle\rho|Y^2|\rho\rangle = \langle X^2\rangle\langle Y^2\rangle \\ \implies 4\langle X^2\rangle\langle Y^2\rangle &\geq 4|\langle\rho|XY|\rho\rangle|^2 = |\langle\rho|[X,Y]|\rho\rangle|^2 + |\langle\rho|\{X,Y\}|\rho\rangle|^2 \geq |\langle\rho|[X,Y]|\rho\rangle|^2\end{aligned}$$

Now take $X = A - \langle A\rangle, Y = B - \langle B\rangle$ in the above. Check that $\langle\rho|[X,Y]|\rho\rangle = \langle\rho|[A,B]|\rho\rangle$. So we finally have

$$\Delta(A)\cdot\Delta(B) \geq \frac{1}{2}\langle [A, B]\rangle$$

□

4 Quantum Gates

Definition 4.1. An $n-$*qubit quantum gate* is a unitary operator on $(\mathbb{C}^2)^{\otimes n}$ (or unitary matrix, considering canonical basis).

4.1 $1-$qubit quantum gates

4.1.1 Pauli gates

The NOT gate: $\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\sigma_1 |0\rangle = |1\rangle, \sigma_1 |1\rangle = |0\rangle$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

4.1.2 Hadamard gate

$\mathcal{H} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$|+\rangle := \mathcal{H} |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \qquad |-\rangle := \mathcal{H} |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Exercise 7
Verify that $\mathcal{H}^{\otimes n} |\boldsymbol{x}\rangle = \frac{1}{2^{n/2}} \sum_{\boldsymbol{y} \in \{0,1\}^n} (-1)^{\langle \boldsymbol{x} | \boldsymbol{y} \rangle} |\boldsymbol{y}\rangle$

4.1.3 Phase gate

$S := |0\rangle + i |1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

4.1.4 $\frac{\pi}{8}-$gate

$T := e^{i\frac{\pi}{8}} \begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$

4.2 $2-$qubit quantum gates

4.2.1 Controlled *NOT* gate

This gate reads the first qubit and performs the NOT operation on the second qubit. That is, we have the following transformations:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle \end{aligned}$$

Let π_1, π_2 be the projection operators $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (respectively) on $\mathbb{C}^2$. Then the $CNOT$ gate can be defined as

$$CNOT := \pi_1 \otimes \mathbf{1} + \pi_2 \otimes \sigma_x$$

We can also define another $CNOT$ gate which functions in exactly the opposite way: The control qubit is the second bit and the NOT is performed on the first bit. So define

$$C'NOT := \mathbf{1} \otimes \pi_1 + \sigma_x \otimes \pi_2$$

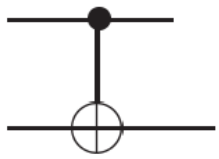

Figure 1: $CNOT$ gate (with control in the top channel)

The matrix forms look like

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C'NOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example. The column matrix forms of the vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle \in H \otimes H$ would be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ respectively. So, applying $CNOT$ on the vector $(a|0\rangle + b|1\rangle) \otimes |0\rangle$ gives

$$(CNOT)(a|00\rangle + b|10\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ b \end{bmatrix} = a|00\rangle + b|11\rangle$$

4.2.2 Controlled U gate

This gate is similar to the controlled NOT gate. The first qubit acts as the control bit, and the unitary operation U is performed on the second qubit. That is, we have the following transformations:

$$\begin{aligned} |00\rangle &\mapsto |0\rangle \otimes |0\rangle \\ |01\rangle &\mapsto |0\rangle \otimes |1\rangle \\ |10\rangle &\mapsto |1\rangle \otimes (U|0\rangle) \\ |11\rangle &\mapsto |1\rangle \otimes (U|1\rangle) \end{aligned}$$

The gate can be defined by

$$CU := \pi_1 \otimes \mathbf{1} + \pi_2 \otimes U$$

Similarly we can define

$$C'U := \mathbf{1} \otimes \pi_1 + U \otimes \pi_2$$

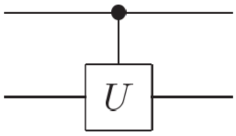

Figure 2: Controlled U gate (with control in the top channel)

The matrix form looks like

$$CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \qquad C'U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

where $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Remark 4.2. The $CNOT$ and $C'NOT$ are just the cases when $U = \sigma_x$.

4.2.3 SWAP gate

The $SWAP$ gate does as you might expect: it swaps the two qubits. So the transformations are as follows:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |10\rangle \\ |10\rangle &\mapsto |01\rangle \\ |11\rangle &\mapsto |11\rangle \end{aligned}$$

It is relatively easier to construct the matrix first

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXERCISE 8
Verify that $SWAP = (CNOT)(C'NOT)(CNOT)$

The above exercise helps us to define

$$SWAP := \pi_1 \otimes \pi_1 + \pi_2 \otimes \pi_1 + (X\pi_1) \otimes (X\pi_2) + (X\pi_2) \otimes (X\pi_1)$$

4.3 3−qubit quantum gates

4.3.1 Toffoli Gate ($CCNOT$)

$$CCNOT = \begin{bmatrix} \mathbf{1}_6 & \mathbf{0}_{6\times 2} \\ \mathbf{0}_{2\times 6} & X \end{bmatrix}$$

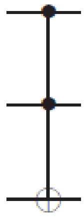

Figure 3: The Toffoli Gate (or $CCNOT$ gate) with control in the first two qubits

4.4 Preparing Bell states

Bell states can be prepared by the action of $\mathcal{H} \otimes \mathbf{1}$ followed by $CNOT$ on the standard bases of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

$$\begin{aligned}
(CNOT)(\mathcal{H} \otimes \mathbf{1}) |00\rangle =& CNOT \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle \right) \\
=& CNOT \left(\frac{1}{\sqrt{2}} (|00\rangle + |01\rangle) \right) \\
=& \frac{1}{\sqrt{2}} (|00\rangle + |01\rangle)
\end{aligned}$$