QUANTUM COMPUTATION

Lecture 2

Nilava Metya
nilavam@cmi.ac.in

19 July 2020

Whether mentioned or not, we will assume H to be a finite dimensional Hilbert space over $\mathbb{C}$.

1 Notation

In quantum computation, we generally use the finite dimensional Hilbert space $H = \mathbb{C}^n$. Since this has a canonical basis, say $(\boldsymbol{e}_i)_{i=1}^n$, we can freely talk about a vector or linear operator interchangibly with their matrix forms. And this gives the natural isomorphism $\sum_{i=1}^{n} v_i \boldsymbol{e}_i \mapsto \begin{bmatrix} v_1^* & \dots & v_n^* \end{bmatrix}$, thus H and H^* are conjugate linearly isomorphic. The image of a vector ξ, under the above isomorphism, is called its dual and denoted by ξ^* in mathematics. However for the sake of quantum mechanics, we will denote the vector by $|\xi\rangle$ (read as *ket of xi*) and its dual is denoted by $\langle\xi|$ (read as *bra of xi*), that is, $\langle\xi|^* = |\xi\rangle$.This is Dirac's bra-ket notation.
Also for an indexed subset $\{\xi_\alpha\}_{\alpha\in A}$ of H we will simply write $|\alpha\rangle$ instead of $|\xi_\alpha\rangle$.

Definition 1.1. Let H be a Hilbert space (not necessarily finite dimensional). Any vector in H is denoted by $|\xi\rangle$ and any vector in H^* is denoted by $\langle\eta|$.

Definition 1.2. For Hilbert space H, the action of $\langle\eta| \in H^*$ on $|\xi\rangle \in H$ is denoted by $\langle\eta|\xi\rangle$.

Definition 1.3. Let H_1, H_2 be Hilbert spaces. For $\langle\eta| \in H_1^*, |\xi\rangle \in H_2$, the operator $|\xi\rangle\langle\eta| \in \mathrm{Hom}(H_1, H_2)$ is a rank one linear operator defined as $|\xi\rangle\langle\eta| \left(|\zeta\rangle\right) = \langle\eta|\zeta\rangle \, |\xi\rangle$.

Proposition 1.4. *Let* $\{|i\rangle\}_{i=1}^n$ *be an orthonormal basis of* H*. Then* $\sum_i |i\rangle\langle i| = I$

Proposition 1.5. *Let* $\langle\eta| \in H_1^*, |\xi\rangle \in H_2, A \in End(H_2)$*, where* $H_1.H_2$ *are finite dimensional Hilbert spaces. Then*

1. $\left(|\xi\rangle\langle\eta|\right)^* = |\eta\rangle\langle\xi|$
2. $\left(A\,|\xi\rangle\right)^* = \langle\xi|\,A^*$
3. $Tr\left(|\xi\rangle\langle\eta|\right) = \langle\eta|\xi\rangle$

Definition 1.6. For $|\xi_i\rangle \in H$, we denote

$$\begin{aligned} |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_n\rangle &=: |\xi_1\rangle\,|\xi_2\rangle \dots |\xi_n\rangle \\ &=: |\xi_1\xi_2\dots\xi_n\rangle \end{aligned}$$

If $H = \mathbb{C}^2$, each $\xi_i \in \{0,1\}$. In that case, we denote $|\boldsymbol{\xi}\rangle := |\xi_1\xi_2\dots\xi_n\rangle$ where $\boldsymbol{\xi} = \sum_{i=1}^n 2^{n-i}\xi_i$.

Definition 1.7. For $|\xi\rangle \in H$, we denote $|\xi\rangle^{\otimes k} := \underbrace{|\xi\rangle \otimes \cdots \otimes |\xi\rangle}_{k \text{ times}}$

2 States and events

We will be making general definitions. The $2D$ Hilbert space $\mathbb{C}^2$ (and $M_2(\mathbb{C})$) will be extensively used for examples.

2.1 Introduction

Definition 2.1. A *projection* is a linear operator $T \in \mathrm{End}(H)$ such that $T = T^* = T^2$.

Exercise 1
Check the following properties of the tensor product:

1. $(A \otimes B)^* = A^* \otimes B^*$
2. $(A \otimes B)^t = A^t \otimes B^t$
3. Tensor product of two unitary operators is unitary
4. Tensor product of two Hermitian operators is Hermitian
5. Tensor product of two non-negative operators is non-negative
6. Tensor product of two projections is a projection

Definition 2.2. A *state* is a linear functional $\varphi : \mathrm{End}(H) \to \mathbb{C}$ such that $\varphi(A^*A) \geq 0 \; \forall \; A \in \mathrm{End}(H)$ and $\varphi(\mathbf{1}) = 1$ ($\mathbf{1}$ is the identity operator).

Example. Fix $\xi \in H, \|\xi\| = 1$, the linear functional φ given by $A \overset{\varphi}{\mapsto} \langle \xi | A\xi \rangle$ is a state.

Definition 2.3. Let $\xi \in H, \|\xi\| = 1$. The *vector state* corresponding to ξ is the linear operator $\varphi \in \mathrm{End}(H)^*$ given by $A \overset{\varphi}{\mapsto} \langle \xi | A\xi \rangle \;\; \forall \; A \in \mathrm{End}(H)$.

Definition 2.4. A (quantum mechanical) *event* is a projection in $\mathrm{End}(H)$, where H is any Hilbert space.

Definition 2.5. Two events $E, F \in \mathrm{End}(H)$ are said to be *compatible* iff they commute, that is $EF = FE$.

Proposition 2.6. *Let $E \in M_2(\mathbb{C})$ be an event. Then $\exists \; \boldsymbol{a} = (x, y, z) \in S^2 \subseteq \mathbb{R}^3$ such that the event is given by* $E = E_{\boldsymbol{a}} = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$.

Definition 2.7. We define the following matrices for $(x, y, z) \in \mathbb{R}^3$:

$$\sigma(x, y, z) = \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}$$
$$\sigma_x = \sigma_1 = \sigma(1, 0, 0)$$
$$\sigma_y = \sigma_2 = \sigma(0, 1, 0)$$
$$\sigma_z = \sigma_3 = \sigma(0, 0, 1)$$

σ_i are called the *Pauli matrices*.

Exercise 2
Let $\mathbf{1}$ be the identity matrix of order 2 and $\times$ be the usual cross product on $\mathbb{R}^3$. In this exercise we talk about events in $M_2(\mathbb{C})$. Verify the following:

1. $E_{\boldsymbol{a}} = \frac{1}{2}(\mathbf{1} + \sigma(\boldsymbol{a}))$ for $\boldsymbol{a} \in S^2$.
2. $\sigma(\boldsymbol{a})\sigma(\boldsymbol{b}) = \langle a|b\rangle \, \mathbf{1} + i\sigma(\boldsymbol{a} \times \boldsymbol{b})$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.
3. $E_{\boldsymbol{a}} + E_{-\boldsymbol{a}} = \mathbf{1}, E_{\boldsymbol{a}}E_{-\boldsymbol{a}} = 0$ for $\boldsymbol{a} \in S^2$.
4. Let $\boldsymbol{a}, \boldsymbol{b} \in S^2$. $E_{\boldsymbol{a}}$ and $E_{\boldsymbol{b}}$ are compatible iff $\boldsymbol{a} = \pm\boldsymbol{b}$.

Exercise 3
Do you see the real projective plane $\mathbb{RP}^2$ from the previous exercise?

Proposition 2.8. *$\langle\cdot|\cdot\rangle_{\mathrm{Tr}} : End(H) \times End(H) \to \mathbb{C}$ given by $\langle T|S\rangle_{\mathrm{Tr}} := \mathrm{Tr}(T^*S)$ is an inner product.*

Theorem 2.9. *Let $\varphi : End(H) \to \mathbb{C}$ be a state. Then $\exists!\ \rho \in \{\chi \in End(H) \mid \chi \geq 0, \mathrm{Tr}(\chi) = 1\}$ such that $\varphi(A) = \mathrm{Tr}(\rho A)\ \forall\ A \in End(H)$.*

Definition 2.10. The probability of a (quantum mechanical) event E with respect to state φ is $\mathrm{Prob}_\varphi(E) := \varphi(E)$

Proposition 2.11. *Let $\rho \in M_2(\mathbb{C})$ so that $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$ (non-negative definite). Then:*

1. *ρ is given by $\rho_{\boldsymbol{a}} = \frac{1}{2}(1 + \sigma(\boldsymbol{a}))$ for some $\boldsymbol{a} = (x, y, z) \in \mathbb{R}^3, \|\boldsymbol{a}\| \leq 1$.*
2. *$Prob_\varphi(E_{\boldsymbol{b}})\Big(= \mathrm{Tr}(\rho_{\boldsymbol{a}}E_{\boldsymbol{b}})\Big) = \frac{1}{2}(1 + \langle \boldsymbol{a}|\boldsymbol{b}\rangle)$ where φ is the state corresponding to $\rho_{\boldsymbol{a}}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3, \|\boldsymbol{a}\| \leq 1$.*

Definition 2.12. $X \in \mathrm{End}(H)$ is said to be an *observable* iff $X^* = X$.

2.2 von Neumann's collapse Postulate

Let $X \in \mathrm{End}(H)$ be a self adjoint operator (observable), and $\rho \in \mathrm{End}(H)$ a state with respect to which all measurements will be made. Let the spectral decomposition (with respect to a given basis) of X be $X = \sum_{\lambda \in \sigma(X)} \lambda E_\lambda$. When X is measured with respect to ρ, it realizes an eigenvalue λ with probability $\mathrm{Tr}(\rho E_\lambda)$ and the state collapses to $\tilde{\rho} = \dfrac{E_\lambda \rho E_\lambda}{\mathrm{Tr}(\rho E_\lambda)}$.

3 Actual Quantum Computation (What we require)

Definition 3.1. A *qubit* is a vector state in $\mathbb{C}^2$.

Remark 3.2. Since a qubit ψ is a vector state, $\psi = |\xi\rangle\langle\xi|$ for some $|\xi\rangle \in H, \|\xi\| = 1$. We will refer to $|\xi\rangle$ as the qubit instead of ψ. We will denote $|0\rangle = (0, 1), |\xi\rangle = (1, 0) \in \mathbb{C}^2$. So a qubit would look like $a\,|0\rangle + b\,|1\rangle\,, |a|^2 + |b|^2 = 1, a, b \in \mathbb{C}$.

Remark 3.3. If a basis is not mentioned, *measuring* a qubit $|\xi\rangle$ means measuring the observable $X = 0\,|0\rangle\langle 0| + 1\,|1\rangle\langle 1|$ with respect to the state $|\xi\rangle\langle\xi|$.

Exercise 4
Measuring $\xi = a|0\rangle + b|1\rangle$ gives value 0 with probability $|a|^2$ and value 1 with probability $|b|^2$. Post measurement, $|\xi\rangle$ collapses to $\left|\tilde{\xi}\right\rangle = |0\rangle$ or $\left|\tilde{\xi}\right\rangle = |1\rangle$ respectively.

Definition 3.4. A *quantum gate* is a unitary operator in $\mathrm{End}(H)$.

Remark 3.5.

- $1-$qubit quantum gate is a unitary operator in $\mathrm{End}(\mathbb{C}^2)$
- $n-$qubit quantum gate is a unitary operator in $\mathrm{End}((\mathbb{C}^2)^{\otimes n})$
- An $n-$qubit quantum gate $U_1 \otimes \cdots \otimes U_n$ acts as

$$(\otimes_{i=1}^n U_i)\,(\otimes_{i=1}^n |\xi_i\rangle) := \otimes_{i=1}^n |U_i \xi_i\rangle$$

4 Quantum entanglement

Definition 4.1. Let H_1, H_2 be two Hilbert spaces.
$|\xi\rangle \in H_1 \otimes H_2$ is a *product state* iff $\exists\ |\xi_1\rangle \in H_1, |\xi_2\rangle \in H_2$ such that $|\xi\rangle = |\xi_1\rangle \otimes |\xi_2\rangle$.
$|\xi\rangle \in H_1 \otimes H_2$ is an *entangled state* iff $|\xi\rangle$ is not a product state.

Example. The following are called the Bell states in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
- $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

The Bell states are entangled states. We will check that (the definition) for $|\Phi^+\rangle$. Suppose that $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ for some $a, b, c, d \in \mathbb{C}$.
This would give us $ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
On one hand $ac = bd = \frac{1}{\sqrt{2}} \implies abcd = \frac{1}{2}$ and on the other hand, $ad = bc = 0 \implies abcd = 0$ which is absurd!
The following exercise brings out the physical meaning of entangled states.

Exercise 5
Let $|\alpha\rangle, |\beta\rangle$ be an orthonormal basis of $H = \mathbb{C}^2$ and consider the observable $X = 0\,|\alpha\rangle\langle\alpha| + 1\,|\beta\rangle\langle\beta|$. Measuring $X \otimes \mathbf{1}$ with respect to the Bell state $|\Phi^+\rangle$ forces the second qubit to collapse to the conjugate of same state as the first qubit. Measuring with respect to $|\Psi^-\rangle$ does the opposite. Try for $|\Phi^-\rangle$ and $|\Psi^+\rangle$.