{Lecture 3}

Random Field Ising Model (RFIM)

Consider $H_N(\vec{s}, \vec{h}) = -\frac{N}{2}\left(\frac{\sum_i s_i}{N}\right)^2 - \sum_i h_i s_i$,

$s_i = \{-1, +1\}$

where $h_i \sim N(0, \Delta)$ random fields

As before, we're interested in the $N \to \infty$ limit.

Focus on $\underbrace{\lim_{N\to\infty} \frac{1}{N} \log Z_N(\vec{h})}_{=\Phi(\beta,\Delta), \text{ as explained below}}$, where

$\vec{h} = (h_1, \dots, h_N)$ and $Z_N(\vec{h}) = \sum_{\{\vec{s}\}} e^{-\beta H_N(\vec{s}, \vec{h})}$.

$\frac{1}{N} \log Z_N$ is a random variable which converges to its mean in the $N \to \infty$ limit. Then $\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, \vec{h}) \; (=)$

"self-averaging"

make T-dependence explicit

$$(=) \lim_{N\to\infty} \frac{1}{N} E_{\vec{h}}[\log Z_N(\beta, \vec{h})] \equiv$$

$$\equiv \Phi(\beta, \Delta).$$

We will compute $\phi(\beta, \Delta)$ with the <u>replica</u> <u>method</u>:

Consider $Z^n = e^{n \log Z} \underset{n \ll 1}{=} 1 + n \log Z + o(n)$, or

$$\underbrace{\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}.}$$

'replica trick'

Then $E[\log Z] = E\left[\lim_{n\to 0} \frac{Z^n - 1}{n}\right] =$

$$= \lim_{n \to 0} \frac{E[Z^n] - 1}{n}.$$

$n \in \mathbb{Z} \Rightarrow n \in \mathbb{R} \Rightarrow$
$\Rightarrow n \to 0$

$E[Z^n]$ is easier to compute than $E[\log Z]$, for $n \in \mathbb{Z}$.

<u>Replicated</u> <u>partition</u> <u>function</u>:

$$Z^n = \sum_{\{\vec{S}^{(1)}\}} \cdots \sum_{\{\vec{S}^{(n)}\}} e^{\beta \sum_{\alpha=1}^{n} \left[\frac{N}{2} \left(\frac{\sum_i S_i^{(\alpha)}}{N} \right)^2 + \sum_i h_i S_i^{(\alpha)} \right]}.$$

Next,

$$E_{\vec{h}}[Z^n] = E_{\vec{h}}\left[\sum_{\{\vec{S}^{(\alpha)}\}_{\alpha=1}^{n}} e^{\beta \frac{N}{2} \sum_{\alpha=1}^{n} \left(\frac{\sum_i S_i^{(\alpha)}}{N} \right)^2 + \beta \sum_\alpha \sum_i h_i S_i^{(\alpha)}} \right] \boxed{=}$$

$$\boxed{=} N^n \mathbb{E}_{\vec h}\Big[\sum_{\{\vec S^{(\alpha)}\}_{\alpha=1}^n}\Big(\int dm_1 \dots dm_n \times$$

$$\times\, \delta\Big(\sum_i S_i^{(1)} - N m_1\Big)\dots\delta\Big(\sum_i S_i^{(n)} - N m_n\Big)\times$$

$$\times\, e^{\beta\frac{N}{2}m_1^2}\dots e^{\beta\frac{N}{2}m_n^2}\Big)e^{\beta\sum_\alpha\sum_i h_i S_i^{(\alpha)}}\Big] \diamondsuit$$

Recall that $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x}$.

$$\diamondsuit \left(\frac{N}{2\pi}\right)^n \mathbb{E}_{\vec h}\Big[\sum_{\{\vec S^{(\alpha)}\}_{\alpha=1}^n} e^{\beta\sum_\alpha\sum_i h_i S_i^{(\alpha)}}\times$$

$$\times\Big(\prod_{\alpha=1}^n\int dm_\alpha\, d\lambda_\alpha\, e^{i\lambda_\alpha(\sum_i S_i^{(\alpha)} - N m_\alpha)}e^{\beta\frac{N}{2}m_\alpha^2}\Big) =$$

$$i\lambda_\alpha = \hat m_\alpha \Rightarrow d\lambda_\alpha = \frac{d\hat m_\alpha}{i}$$

$$= \left(\frac{N}{2\pi i}\right)^n \mathbb{E}_{\vec h}\Big[\sum_{\{\vec S^{(\alpha)}\}_{\alpha=1}^n} e^{\beta\sum_\alpha\sum_i h_i S_i^{(\alpha)}}\Big(\prod_{\alpha=1}^n\int dm_\alpha\, d\hat m_\alpha \times$$

$$\times\, e^{\hat m_\alpha(\sum_i S_i^{(\alpha)} - N m_\alpha)}e^{\beta\frac{N}{2}m_\alpha^2}\Big)\Big] \sim$$

$$\sim \int\dots\int\Big(\prod_{\alpha=1}^n dm_\alpha\, d\hat m_\alpha\Big)e^{\beta\frac{N}{2}\sum_\alpha m_\alpha^2 - N\sum_\alpha \hat m_\alpha m_\alpha}\times$$

$$\times\, \mathbb{E}_{\vec h}\Big[\sum_{\{\vec S^{(\alpha)}\}_{\alpha=1}^n}\underbrace{e^{\sum_\alpha \hat m_\alpha(\sum_i S_i^{(\alpha)}) + \beta\sum_\alpha\sum_i h_i S_i^{(\alpha)}}}_{\prod_\alpha\prod_i e^{\hat m_\alpha S_i^{(\alpha)} + \beta h_i S_i^{(\alpha)}}}\Big] \circledcirc$$

$$\stackrel{\circledcirc}{=} \int\!\dots\!\int \left(\prod_\alpha dm_\alpha d\hat{m}_\alpha\right) e^{\beta \frac{N}{2}\sum_\alpha m_\alpha^2 - N\sum_\alpha \hat{m}_\alpha m_\alpha} \times$$

$$\times\, e^{N \log\left(E_h\left[\prod_\alpha 2\cosh(\beta h + \hat{m}_\alpha)\right]\right)}$$

Indeed,

$$E_{\vec{h}}[\dots] = E_{\vec{h}}\Big[\prod_i \prod_\alpha \underbrace{\sum_{S_i^{(\alpha)}=\pm 1} e^{\hat{m}_\alpha S_i^{(\alpha)} + \beta h_i S_i^{(\alpha)}}}_{2\cosh(\beta h_i + \hat{m}_\alpha)}\Big] =$$

$$= E_{\vec{h}}\Big[\prod_i \Big(\prod_\alpha 2\cosh(\beta h_i + \hat{m}_\alpha)\Big)\Big] =$$

$$= \Big\{\underset{\substack{\uparrow\\ \text{single } h!}}{E_h}\Big[\prod_\alpha 2\cosh(\beta h + \hat{m}_\alpha)\Big]\Big\}^N.$$

So, $E_{\vec{h}}[Z^n] \sim \int dm_1 d\hat{m}_1 \dots dm_n d\hat{m}_n \times$

$$\times\, e^{N\left\{\frac{\beta}{2}\sum_\alpha m_\alpha^2 - \sum_\alpha m_\alpha \hat{m}_\alpha + \log\left(E_h\left[\prod_\alpha 2\cosh(\beta h + \hat{m}_\alpha)\right]\right)\right\}}.$$

<u>assume</u> $\begin{cases} m_\alpha = m \\ \hat{m}_\alpha = \hat{m} \end{cases} \quad \forall \alpha$

[<u>replica</u> <u>symmetry</u> <u>ansatz</u>.]
(RS)

Then $E_{\vec{h}}[Z^n] \sim \int dm\, d\hat{m}\; e^{N\left\{\frac{\beta}{2} n m^2 - n m \hat{m} + \log\left(E_h\left[2^n \cosh^n(\beta h + \hat{m})\right]\right)\right\}}$

use saddle-point

Finally,

$$\phi(\beta, \Delta) = \underset{\substack{\uparrow \\ \text{replica trick}}}{\lim_{N\to\infty}} \frac{1}{N} \lim_{n\to 0} \frac{E_{\vec{h}}[Z^n(\beta,\vec{h})] - 1}{n} \overset{\text{swap } \lim_{N\to\infty} \,\&\, \lim_{n\to 0} \text{ (not rigorous)}}{=}$$

$$= \lim_{n\to 0} \frac{1}{n} \lim_{N\to\infty} \frac{E_{\vec{h}}[Z^n] - 1}{N}.$$

swap $\lim_{N\to\infty}$ & $\lim_{n\to 0}$ (<u>not</u> rigorous)

$$\frac{1}{N}\log(E_{\vec{h}}[Z^n]) \xrightarrow[N\to\infty]{} \max_{m,\hat{m}} \Big\{ \frac{\beta}{2} n m^2 - n m \hat{m} + \log(E_h[2^n \cosh^n(\beta h + \hat{m})]) \Big\}.$$

as $n \to 0$, $E_{\vec{h}}[Z^n] \to 1$ from above and

$$\log(E_{\vec{h}}[Z^n]) \to \underline{\underline{E_{\vec{h}}[Z^n] - 1}}.$$

Taylor expansion of $\log(x)$

So,

$$\phi(\beta,\Delta) = \underbrace{\lim_{n\to 0} \frac{1}{n} \max_{m,\hat{m}} \Big\{ \frac{\beta}{2} n m^2 - n m\hat{m} + \log(E_h[2^n \cosh^n(\beta h + \hat{m})]) \Big\}}$$

Next, note that $E[X^n] = E[e^{n\log X}] \underset{n \ll 1}{\approx}$

$$\approx E[1 + n\log X] = 1 + nE[\log X] \approx$$

$$\approx e^{nE[\log X]}, \text{ or}$$

$$\log E[X^n] \simeq n\, E[\log X].$$

E can be taken in & out of the log.

Then $$\varphi(\beta,\Delta) = \max_{m,\hat{m}} \left\{ \frac{\beta}{2} m^2 - m\hat{m} + E_h[\log(2\cosh(\beta h + \hat{m}))] \right\}$$

Saddle points:

$$\frac{\partial}{\partial m}\{\ldots\} = \beta m - \hat{m} = 0 \Rightarrow \underline{\underline{\hat{m} = \beta m}}$$

Then $$\varphi_{RS}(m,\beta,\Delta) \equiv -\frac{\beta}{2} m^2 + E_h[\log(2\cosh[\beta(h+m)])].$$

So, $$\varphi(\beta,\Delta) = \max_m \varphi_{RS}(m,\beta,\Delta) = \varphi_{RS}(m^*,\beta,\Delta),$$

where

$$\frac{\partial}{\partial m}\varphi_{RS} = -\beta m + E_h\left[\frac{2\sinh[\beta(h+m)]}{2\cosh[\beta(h+m)]}\right]\beta = 0, \text{ or}$$

$$m = E_h[\tanh[\beta(h+m)]] =$$

$$= \underline{\underline{\int_{-\infty}^{\infty} dh\, \frac{e^{-h^2/2\Delta}}{\sqrt{2\pi\Delta}} \tanh[\beta(h+m)]}}.$$

self-consistent eq'n for m,
need to solve it to find m^*
and thus $\varphi(\beta,\Delta)$.

Recall that the average energy per spin is given by

$$\langle e \rangle = -\frac{\partial \Phi(\beta,\Delta)}{\partial \beta} = \frac{(m^*)^2}{2} - E_h\left[(h+m^*)\tanh[\beta(h+m^*)]\right]$$

as $T \to 0$ $(\beta \to \infty)$,

$\tanh[\beta(h+m^*)] \to \mathrm{sgn}(h+m^*)$, so that

$$\langle e \rangle \xrightarrow[T\to 0]{} \frac{(m^*)^2}{2} - E_h\left[(h+m^*)\,\mathrm{sgn}(h+m^*)\right]$$

min E of the system