Representation theory of Lie algebras

Nilava Metya

PROMYS 2021

1 Motivation

1.1 Introduction

We all know $M(n, \mathbb{R})$ and have seen this in many different ways.

1. Vector space: we can add matrices and multiply by complex scalars.
2. Ring: we can multiply square matrices.
3. Geometry: You can think it as literally $\mathbb{R}^{n^2}$, just with the linear 'look' of $\mathbb{R}^{n^2}$ being changed to a tabular form. It's possible to measure distances. Some examples:
   - $\|M\| = \sum\limits_{1 \leq i,j \leq n} |M_{ij}|$
   - $\|M\|_{p,q} = \left\{ \sum\limits_{j=1}^{n} \left\{ \sum\limits_{i=n}^{n} |M_{ij}|^p \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$
4. Manifold: umm...

1.2 What is a manifold?

Think of placing 'charts' on a surface, so that there is no roughness or fold-lines on the chart - in other words, we should be able to pick up a small region on the surface and *smoothly* deform it into $\mathbb{R}^2$ in a one-one way. And these charts are 'stitched' smoothly.
Well, for a manifold, you would allow this to happen for $\mathbb{R}^n$ for any $n \geq 1$, not just $\mathbb{R}^2$. Since we know how to do calculus on $\mathbb{R}^n$, and we can easily (and smoothly) transition between charts and $\mathbb{R}^n$, we know how to do calculus in any manifold.

Definition 1.1 (Submanifolds of $\mathbb{R}^n$)**.** A subset $M \subseteq \mathbb{R}^n$ is said to be an m-dimensional **submanifold** of $\mathbb{R}^n$ if $\forall \boldsymbol{x} \in M, \exists W \underset{\text{open}}{\subset} \mathbb{R}^n$ containing $\boldsymbol{x}$ such that $W \cap M$ is diffeomorphic to some $U \underset{\text{open}}{\subset} \mathbb{R}^m$.
The diffeomorphism $\psi : U \to W \cap M$ is called a **parameterization**.

We'll often denote by φ the inverse of the above ψ. We'll often index the above ψ, φ with the point x.
Let's look at some ways to build new manifolds from known ones. We indeed look at some lemmas.

Proposition 1.2. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension m and $M' \underset{\text{open}}{\subset} M$. Then M' is a manifold of dimension m.

Proof. Fix $x \in M' \subseteq M$. Then $\varphi_x : W_x \cap M \to U_x$ is a diffeomorphism, where the symbols mean as in the above definition. The restriction of φ_x to M' does the job. ∎

Example. We can note that $M(n, \mathbb{R}) = \mathbb{R}^{n^2}$ is a manifold and $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, and thus $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open sebset of $M_n(\mathbb{R})$. Conclude that $GL_n(\mathbb{R})$ is an n^2 dimensional manifold.

Theorem 1.3 (Implicit function theorem)**.** Let $m < n, \Omega \underset{\text{open}}{\subset} \mathbb{R}^n, \varphi \in \mathscr{C}^1(\Omega, \mathbb{R}^m)$. Let $M_{\boldsymbol{c}} = \varphi^{-1}(\boldsymbol{c})$ be non-empty such that $\mathcal{J}\varphi(\boldsymbol{x})$ has full rank (namely, m) $\forall \boldsymbol{x} \in M_{\boldsymbol{c}}$. Then $M_{\boldsymbol{c}}$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$.

We are not going to prove the above theorem. However this will be useful in building manifolds.

Example $(\varphi(x, y) = y - e^x)$**.** $\varphi \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R})$. So $n = 2, m = 1$ here. $M_0 = \{(x, e^x) : x \in \mathbb{R}\}$.
$\mathcal{J}\varphi(x, y) = \begin{bmatrix} -e^x & 1 \end{bmatrix}$. This has rank 1 always. The graph of this function is a manifold of dimension $n - m = 1$.
A manifold having dimension 1 (that is, parameterized by only one variable) simply means that it is a 'curve'.

Example $(\varphi(x, y, z) = x^2 + y^2 + z^2)$**.** $\varphi \in \mathscr{C}^1(\mathbb{R}^3, \mathbb{R})$. So $n = 3, m = 1$ here. $M_1 = \mathbb{S}^2$.
$\mathcal{J}\varphi(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$. This has rank 1 always. The graph of this function is a manifold of dimension $n - m = 2$.
A manifold having dimension 2 (that is, parameterized by two variables) means that it is a 'surface'.

Example (Special mention: $\varphi(\boldsymbol{x}) = 0$)**.** $\varphi \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}^0)$. So $m = 0$ here. $M_0 = \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}) = 0\} = \mathbb{R}^n$. $\mathcal{J}\varphi(x, y, z) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$. This has rank 0 (full!) always. The graph of this function is a manifold of dimension $n - m = n$.
Here, the manifold is characterized by n independent parameters.

1.3 Lie Groups

A Lie group is a group G which is also a manifold, where $g \mapsto g^{-1}$ and $(g, h) \mapsto gh$ are smooth.
We saw above that $GL(n, \mathbb{R})$ is a Lie group. From now, we will refer to e_{ij} as the matrix having 1 at position (i, j) and 0 elsewhere (dimension of the matrix will be clear form the context).

Example $(SL(n, \mathbb{R}))$**.** Consider $\det : M(n, \mathbb{R}) \to \mathbb{R}$. I invite the reader to prove that $D_H \det(A) = \operatorname{Tr}(\operatorname{adj}(A)H)$ as an exercise. Here D_H is the directional derivative along H. Then $A \in SL_n(\mathbb{R}) = \det^{-1}(1) \implies D\det(A) = \sum\limits_{i,j} D_{e_{ij}}(A)e_{ij}$ has rank 1. So, $SL(n, \mathbb{R})$ is a manifold of dimension $n^2 - 1$.

Example $(\mathfrak{t}(n, \mathbb{R}))$**.** Define $\mathfrak{t}(n, \mathbb{R})$ to be the set of all $n \times n$ strictly upper triangular matrices with entries from $\mathbb{R}$. This is simply $\mathbb{R}^{\frac{n(n-1)}{2}}$. Let $\mathfrak{n}'(n, \mathbb{R})$ be the set of all $n \times n$ (non-strict) lower triangular matrices with entries from $\mathbb{R}$. This is simply $\mathbb{R}^{\frac{n(n+1)}{2}}$. Define $\pi : M(n, \mathbb{R}) \to \mathfrak{n}'(n, \mathbb{R})$ be given by $\sum\limits_{i,j} a_{ij}e_{ij} \mapsto \sum\limits_{i \geq j} a_{ij}e_{ij}$. Then $\mathfrak{t}(n, \mathbb{R}) = \pi^{-1}(0)$. We can note that the rank of $D\pi$ is $\frac{n(n+1)}{2}$ (full!). This says that $\mathfrak{t}(n, \mathbb{R})$ is a $\frac{n(n-1)}{2}$-dimensional manifold.

Example ($O(n,\mathbb{R})$)**.** Let $\varphi = (A \mapsto A^tA) : M(n,\mathbb{R}) \to M(n,\mathbb{R}), \alpha = \pi \circ \varphi : M(n,\mathbb{R}) \to \mathfrak{n}'(n,\mathbb{R})$ where π is the map in the previous example. We claim that $\varphi^{-1}(I) = \alpha^{-1}(I)$.
Indeed, $\varphi(A) = I \implies \alpha(A) = \pi(I) = I$. And suppose $\alpha(A) = I$ which just means that the elements of A^tA are all 1 on diagonal and all 0 below diagonal. If $i > j$ then $(A^tA)_{ji} = \langle Ae_j, Ae_i, =\rangle \langle Ae_i, Ae_j, =\rangle e_i^tA^tAe_j = (A^tA)_{ij} = 0$. This just means that A^tA has all 0 above the diagonal. It follows that $A^tA = I$.
One can check that $D\alpha(A)$ has rank $\frac{n(n+1)}{2} \forall A \in \alpha^{-1}(I) = \varphi^{-1}(I) = O(n,\mathbb{R})$. It follows that $O(n,\mathbb{R})$ is a $\frac{n(n-1)}{2}$-dimensional manifold.

Example ($SO(n,\mathbb{R})$)**.** This is the (open) connected component of $O(n)$ with all matrices having $\det = 1$, that is, $SO(n,\mathbb{R}) = \det|_{O(n,\mathbb{R})}^{-1}(\mathbb{R} \setminus \{1\})$. It follows that $SO(n,\mathbb{R})$ is a manifold, having the same dimension as $O(n,\mathbb{R})$, that is $\frac{n(n-1)}{2}$.

1.4 Tangent spaces and Derivations

Example. (How to compute tangents?) Let's consider $\varphi(x,y,t) = (x - y, y - t^2)$ along with the level set $M_{(1,0)} = \left\{(x,y,t) : x - y = 1, y = t^2\right\} = \left\{\left(1 + t^2, t^2, t\right) : t \in \mathbb{R}\right\}$. Call this curve $\boldsymbol{\gamma}$. We all know how to compute the tangent at, say, $\boldsymbol{p} = (2,1,-1)$.
We first 'solve' for $m = 2$ coordinates in terms of the other 'free' $n - m = 1$ coordinates: $x = 1 + t^2, y = t^2$. Then $\dot{x} = 2t, \dot{y} = 2t, \dot{t} = 1$ so that the required direction of the 'velocity' at the point $(2,1,-1)$ is just $\boldsymbol{v} = (-2,-2,1)$.
But we want the line to be passing through $\boldsymbol{p}$. So we say that out line is just given by $\{\boldsymbol{p} + \boldsymbol{v}s : s \in \mathbb{R}\}$.
One thing to note is that $\boldsymbol{\gamma}(-1+s) = \boldsymbol{\gamma}(-1) + \dot{\boldsymbol{\gamma}}(-1)s + \cdots$. However for all out purposes (because we want our tangent spaces to be linear spaces!), we'll take $\mathbb{R}\boldsymbol{v}$ to be our tangent space.

1.4.1 Derivatives

Let $\boldsymbol{f} : \Omega \underset{\text{open}}{\subset} \mathbb{R}^m \to \mathbb{R}^n$ be a function and $\boldsymbol{p} \in \Omega$. We say $\boldsymbol{f}$ is differentiable at $\boldsymbol{p}$ if there is a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\|\boldsymbol{f}(\boldsymbol{p}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{p}) - T\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = 0.$$

It can be proven that such a map T, if exists, is unique. We say $T = D\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{f}'(\boldsymbol{p})$ is **the derivative of $\boldsymbol{f}$ at $\boldsymbol{p}$**.
Now, fix a 'direction' $\boldsymbol{v} \in \mathbb{R}^m$. **The directional derivative of $\boldsymbol{f}$ at $\boldsymbol{p}$ along $\boldsymbol{v}$** is given by

$$D_{\boldsymbol{v}}(\boldsymbol{f})(\boldsymbol{p}) = \lim_{t\to 0} \frac{\boldsymbol{f}(\boldsymbol{p}+t\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{v})}{t}$$

It turns out that $D_{\boldsymbol{v}}(\boldsymbol{f})(\boldsymbol{p}) = \left\langle \boldsymbol{v}, \boldsymbol{f}'(\boldsymbol{p})\right\rangle$ is true for sufficiently 'nice' functions.

1.4.2 Tangent spaces

Let M be a manifold. We say $\boldsymbol{\gamma} : (-\varepsilon,\varepsilon) \to M$ is a **smooth path** in M through $\boldsymbol{p}$ in M if $\boldsymbol{\gamma} \in \mathscr{C}^1$ and $\boldsymbol{\gamma}(0) = \boldsymbol{p}$. The **velocity** of $\boldsymbol{\gamma}$ at $\theta \in (-\varepsilon,\varepsilon)$ is $\boldsymbol{v}_{\boldsymbol{\gamma}}(\theta) = \boldsymbol{\gamma}'(\theta)$.
Now fix a point $\boldsymbol{a} \in M$. Consider the smooth paths in M passing through $\boldsymbol{a}$ and consider the linear span of their velocities. We call this the tangent space $T_{\boldsymbol{a}}$ of M at $\boldsymbol{a}$.

1.4.3 Derivations

Recall the multiplication rule for derivatives: $D(uv) = (Du)v + u(Dv)$. We also know that the derivative is a linear. Let's try to generalize this notion.

Let $\boldsymbol{p} \in \mathbb{R}^n$. Consider the set of all pairs $(U, \boldsymbol{f})$ where $U \subseteq \mathbb{R}^n$ is an open neighbourhood of $\boldsymbol{p}$, and $\boldsymbol{f} \in \mathscr{C}^\infty(U, \mathbb{R})$. We define an equivalence relation $\overset{\boldsymbol{p}}{\sim}$ by: $(U, \boldsymbol{f}) \overset{\boldsymbol{p}}{\sim} (V, \boldsymbol{g}) \iff \exists W \underset{\text{open}}{\subset} U \cap V, a \in W$ such that $\boldsymbol{f}|_W = \boldsymbol{s}g|_W$.
If $\boldsymbol{h} \in \mathscr{C}^\infty(A, \mathbb{R})$ with $\boldsymbol{a} \in A \underset{\text{open}}{\subset} \mathbb{R}^n$ then the **germ of $\boldsymbol{h}$** is the equivalence class of $\boldsymbol{h}$ under $\overset{\boldsymbol{a}}{\sim}$. The set of all germs ($\mathscr{C}^\infty$ maps) at a particular point $\boldsymbol{a} \in \mathbb{R}^n$ (i.e., set of all such equivalence classes) is denoted by $\mathscr{C}^\infty_{\boldsymbol{a}}$. This is really an $\mathbb{R}$-algebra.

Definition 1.4 (Derivation)**.** An $\mathbb{R}$-linear map $D : \mathscr{C}^\infty_{\boldsymbol{a}} \to \mathbb{R}$ is said to be a **derivation at $\boldsymbol{a}$** if $D(uv) = (Du) \cdot v(\boldsymbol{a}) + u(\boldsymbol{a}) \cdot (Dv)$.

One just can't avoid noticing that the set of derivations (at a specified point) is itself a vector space.

Example. $D_{i,\boldsymbol{a}} = \left.\dfrac{\partial}{\partial x_i}\right|_{\boldsymbol{a}}$ is a derivation on $\mathscr{C}^\infty_{\boldsymbol{a}}$. These, in fact, form a basis for the set of derivations $\mathrm{Der}_{\boldsymbol{a}}$ at $\boldsymbol{a}$.

1.5 Putting together

Let $V \underset{\text{open}}{\subset} \mathbb{R}^n$ and $\boldsymbol{\gamma}$ be a curve in V through $\boldsymbol{a}$. It can be shown that $T_{\boldsymbol{a}} \cong \mathbb{R}^n$. Now, for a smooth curve $\boldsymbol{\gamma}$ passing through $\boldsymbol{a}$, define

$$D_{\boldsymbol{\gamma},\boldsymbol{a}}(\boldsymbol{f}_{\boldsymbol{a}}) = \left.\frac{d}{dt}(\boldsymbol{f} \circ \boldsymbol{\gamma})(t)\right|_{\boldsymbol{a}} [= \langle \nabla(\boldsymbol{f})(\boldsymbol{a}), \dot{\boldsymbol{\gamma}}(0)\rangle] \,.$$

Note $\boldsymbol{f}'(\boldsymbol{a})$ depends only on the germ of $(U, \boldsymbol{f})$ at $\boldsymbol{a}$. This means $D_{\boldsymbol{\gamma},\boldsymbol{a}}$ depends only on the velocity vector $\dot{\boldsymbol{\gamma}}(0)$. This gives an injective (check!) linear rule $T_{\boldsymbol{a}} \to \mathrm{Der}_{\boldsymbol{a}}$. Since both of these $\mathbb{R}$-vector spaces have dimension n, we can conclude that the above rule is an isomorphism. So, $T_{\boldsymbol{a}} \cong \mathrm{Der}_{\boldsymbol{a}}$.

Lie algebras arise as tangent spaces to Lie groups at the identity. Just like groups study symmetries, Lie algebras study derivations. The above should make it clear that why tangent spaces give arise to the the study of derivations.

2 Lie algebras

2.1 Introduction

Definition 2.1 (Lie algebra)**.** A vector space L over a field F with an operation $[\cdot, \cdot] : L \times L \to L$ (so $(x, y) \mapsto [x, y]$ or $[xy]$) is called a Lie algebra over F if:

1. $[\cdot, \cdot]$ is bilinear.
2. $[xx] = 0 \forall x \in L$.
3. $[x\,[yz]] + [y\,[zx]] + [z\,[xy]] = 0 \forall x, y, z \in L$.

Some small observations:

1. $0 = [x+y, x+y] = [xx] + [xy] + [yx] + [yy] = [xy] + [yx] \implies [xy] = -[yx]$. In fact, this statement is equivalent to the second statement above, whenever char $F \neq 2$.

2. $[x\,[yz]] + [y\,[zx]] + [z\,[xy]] = 0 \iff [x\,[yz]] = -[y\,[zx]] - [z\,[xy]] = [y, -[zx]] + [[xy]\,z] = [y\,[xz]] + [[xy]\,z]$

Definition 2.2. A Lie algebra L is said to be **abelian** if $[LL] = 0$.

Example ($\mathfrak{gl}(V)$)**.** Let V be a finite dimensional ($= n$) F-vector space. Consider $L = \mathrm{End}(V)$, the associative algebra of endomorphisms of V, along with the commutator $[AB] = AB - BA$ for $A, B \in L$. Verify it's a Lie algebra:

1. Bilinearity is clear.

2. $[AA] = AA - AA = 0$.

3. $[A\,[BC]] + [B\,[CA]] + [C\,[AB]] = [A, BC - CB] + [B, CA - AC] + [C, AB - BA] = ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC = 0$.

We shall use $\mathfrak{gl}(V)$ to distinguish the Lie algebra from the older associative algebra $\mathrm{End}(V)$. The notation is $\mathfrak{gl}$ because it turns out to be the tangent space of the Lie group GL at the identity.
$\mathfrak{gl}(n, F)$ will denote the algebra of $n \times n$ matrices. A basis of $\mathfrak{gl}(n, F) = M_n(F)$ is $\left\{e_{ij} = e_i e_j^t\right\}_{1 \leq i,j \leq n}$.

Example ($\mathbb{R}^3$)**.** Endow $L = \mathbb{R}^3$ the usual cross product $(x, y, z) \times (a, b, c) = (yc - zb, za - xc, xb - ya)$. Define $[\boldsymbol{uv}] = \boldsymbol{u} \times \boldsymbol{v}$.
(Exercise!) Turns out that $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{v})\,\boldsymbol{w} - (\boldsymbol{u} \cdot \boldsymbol{w})\,\boldsymbol{v}$.
This gives $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{0}$.
It is not hard to see that $[\cdot, \cdot]$ is bilinear and $[\boldsymbol{uu}] = \boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$.

Example ($\mathfrak{sl}(V)$)**.** Let V be a finite dimensional ($= n$) F-vector space. Consider $L = \mathfrak{sl}(V) = \{T \in \mathfrak{gl}(V) : \mathrm{Tr}(T) = 0\}$, along with the commutator $[AB] = AB - BA$ for $A, B \in L$. To verify it's a Lie algebra, we do exactly what we did in the previous page.
$\mathfrak{sl}(n, F)$ will denote the algebra of $n \times n$ matrices with zero trace.
A basis of $\mathfrak{sl}(n, F)$ is $\{e_{i,j} : i \neq j, 1 \leq i, j \leq n\} \bigcup \{e_{i,i} - e_{i+1,i+1} : 1 \leq i \leq n-1\}$.
Here, $e_{i,j} = e_i e_j^t$.
An example to keep in mind is $\mathfrak{sl}(2, F)$. This will come up later. An ordered basis is

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

2.2 Abstract Lie algebras

Let L be a finite dimensional F-vector space with basis $\{e_i\}_{i=1}^n$. And suppose we know the commutator $[\cdot, \cdot]$ on L (i.e., we know $[e_i, e_j]$). At a more atomic level, if we say that $[e_i, e_j] = \sum_{k=1}^{n} a_{ij}^k e_k$, then we know all the n^3 numbers $\left\{a_{ij}^k : 1 \leq i, j, k \leq n\right\}$. In fact, those a_{ij}^k's, for which $i \geq j$, can be deduced from the others because $[xy] = -[yx]$. You might observe that these are the only numbers which can uniquely determine my commutator (of course, not arbitrary numbers: for example, taking $a_{11}^k = 1$ is absurd).

Proposition 2.3. Consider a set of n^3 numbers $\left\{a_{ij}^k \in F : 1 \le i,j,k \le n\right\}$ (indexed by i,j,k) satisfying

$$a_{ii}^k = a_{ij}^k + a_{ji}^k = 0 \qquad 1 \le i,j,k \le n$$

$$\sum_{k=1}^{n} \left(a_{ij}^k a_{kj}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m\right) = 0 \qquad 1 \le i,j,m \le n$$

uniquely determines the commutator of a Lie algebra.

2.2.1 Algebras of dimension ≤ 2

Let's determine all Lie algebras (upto isomorphism) of dimension ≤ 2.

1. For dimension 1: If we have $L = Fx$ then clearly $[a,b] = 0 \forall a,b \in L$.
2. For dimension 2: Suppose a basis of L is x,y. The commutator of any two vectors is just a multiple of $[x,y]$. We just need to look at $[xy] = \alpha x + \beta y$, because $[xx] = [yy] = 0$.

   We either have $\alpha = \beta = 0$, in which case L is just abelian.
   Otherwise, we define $x' = [xy]$ and let $y' \in L$ be independent of x'. This will give us $[x'y'] = \lambda [x,y] = \lambda x', \lambda \ne 0$. Now take $y'' = \lambda^{-1} y'$ to finally get $[x'y''] = x'$. So, upto isomorphism, there is **atmost** one non-abelian L. We ensure that atleast one such exists:

   Take $F = \mathbb{R}, L = \mathbb{R}^2, [\boldsymbol{u}, \boldsymbol{v}] = \left(\det \begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}\right) \begin{bmatrix}1\\0\end{bmatrix}$. Indeed, letting $x = \boldsymbol{e}_1, y = \boldsymbol{e}_2$, we have $[x,y] = \left(\det \begin{bmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2\end{bmatrix}\right) \boldsymbol{e}_1 = \boldsymbol{e}_1 = x$.

3 Subspaces and maps

Let L, L' be F-Lie algebras with commutators $[\cdot,\cdot], [\cdot,\cdot]'$.
$\varphi : L \to L'$ is said to be a **homomorphism** of Lie algebras if φ is a homomorphism of vector spaces such that $\varphi([x,y]) = [\varphi(x), \varphi(y)]'$.
Further if φ is an isomorphism of vector spaces, then φ is an **isomorphism** of Lie algebras.
A vector subspace $K \subseteq L$ is said to be a **subalgebra** if $x,y \in K \implies [x,y] \in K$.

3.1 Derivations

Note that the Jacobi identity really gives $[x[yz]] = [y[xz]] + [[xy]z]$. For $a \in L$, write $D_a = [a,\cdot]$.
Just for this section let's write ab instead of $[ab]$.
The Jacobi identity gives us: $D_x(yz) = y(D_x z) + (D_x y)z$. This is just a derivation!

Definition 3.1 (F-algebra)**.** An F-vector space $\mathfrak{V}$ is said to be an F-algebra if it comes with a bilinear operation $\mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$. (We do not ask for associativity.)

Definition 3.2 (Derivation)**.** A derivation δ of an F-algebra $\mathfrak{V}$ is a linear map such that $\delta(ab) = a\delta(b) + \delta(a)b$.

The collection of all derivations of $\mathfrak{V}$ is denoted by $\mathrm{Der}(\mathfrak{V})$.
We get back to our notation $[xy]$ and start hating xy.
Let $\mathfrak{V}$ be an associative algebra over F. It is clearly seen that $\mathrm{Der}(\mathfrak{V})$ is a *subset* of $\mathfrak{gl}(\mathfrak{V})$. Indeed, it is something more (exercise!):

1. If $\delta_1, \delta_2 \in \mathrm{Der}(\mathfrak{V})$, then $[\delta_1, \delta_2] \in \mathrm{Der}(\mathfrak{V})$. (Recall that the bracket in $\mathrm{Der}(\mathfrak{V})$ is $[x, y] = xy - yx$.)
2. If $\delta_1, \delta_2 \in \mathrm{Der}(\mathfrak{V}), a \in F$, then $\delta_1 + a\delta_2 \in \mathrm{Der}(\mathfrak{V})$.

$\mathrm{Der}(\mathfrak{V})$ is a subalgebra of $\mathrm{End}(\mathfrak{V})$.
However, (exercise!) the ordinary product of two derivations need not be a derivation.
For a Lie algebra L, we have already see that $D_x \in \mathrm{Der}(L) \forall x \in L$. In literature, D_x is usually written as $\mathrm{ad}\, x$ or ad_x and read as the **adjoint of** x. Since this derivation comes from *inside* the algebra, we call all such $\mathrm{ad}\, x$'s as **inner derivations**. Others are called **outer derivations**.

Example (ad)**.** Recall $\mathfrak{sl}(2, F)$. Take the ordered basis $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
Check that $[xy] = h, [hx] = 2x, [yh] = 2y$. To determine, say, ad_x, it is enough to look at its action on the basis and then use the coefficients to build up the vector w.r.t. the above basis.
For example $\mathrm{ad}_x(x) = (0, 0, 0), \mathrm{ad}_x(h) = (-2, 0, 0), \mathrm{ad}_x(y) = (0, 1, 0)$ in the matrix form. So this will mean that $\mathrm{ad}_x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. After computing for y, h the final result is

$$\mathrm{ad}_x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{ad}_h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathrm{ad}_y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

3.2 Which subspaces allow quotienting?

A subspace I of a Lie algebra L over F is called an **ideal** of L if $[xy] \in L \forall x \in L, y \in I$.

These are the subspaces of our interest which arise as kernels of hommoprhisms. We can note that every ideal is a subalgebra, but not conversely.

Example. $0, L$ are ideals of L.

Example. The center $Z(L) = \{x \in L : [xz] = 0 \forall z \in L\} = \{x \in L : [xL] = 0\}$. Consider the map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ given by $x \mapsto \mathrm{ad}_x$. The kernel of this map is precisely $\{x \in L : \mathrm{ad}_x = 0\} = \{x \in L : [xL] = 0\} = Z(L)$.
This is very special, an example of a **representation**. We will call this the adjoint representation.

Example. The derived algebra $[LL] = \left\{ \sum_{i \in I} [x_i y_i] : I \text{ finite and } x_i, y_i \in L \forall i \in I \right\}$.
This can be realized as a special case of the fact that if I, J are ideals, so is $[IJ]$ (and $I + J$).

3.2.1 Adjoint representation

Lemma 3.3. $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a homomorphism of Lie algebras.

Proof. It's not hard to see that ad is a vector space homomorphism.
Say $x, y \in L$. Then $\mathrm{ad}_{[xy]}(u) = [[xy]\, u] = -[u\, [xy]] = -[[ux]\, y] - [x\, [uy]] = [y\, [ux]] - [x\, [uy]] = [x\, [yu]] - [y\, [xu]] = (\mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x)(u) = [\mathrm{ad} - x, \mathrm{ad}_y](u)$. So $\mathrm{ad}_{[xy]} = [\mathrm{ad}_x, \mathrm{ad}_y]$. ■

Note that we can also treat $(V =)L$ as a 'linear space' over $(K =)L$: The 'scalar' multiplication is (for $x \in K, v \in V$) $x \cdot v = [xv] = \mathrm{ad}_x(v)$.

Definition 3.4 (Simple Lie algebras)**.** A non-abelian Lie algebra L (i.e., $[LL] \neq 0$) is said to be **simple** if it has no nontrivial proper ideals.

3.3 Automorphisms

Let L be a Lie algebra L. An automorphism of L is an isomorphism $L \to L$. The group of automorphisms of L is denoted by $\mathrm{Aut}(L)$.

Definition 3.5 (Exponential map)**.** Let $\delta \in \mathrm{Der}(L)$ be nilpotent, i.e., $\delta^n = 0$ for some n. We define

$$\exp(\delta) = \sum_{i=0}^{n-1} \frac{\delta^i}{i!}$$

Claim 3.5.1. $\delta \in \mathrm{Der}(L)$ and $\delta^k = 0 \implies \exp(\delta) \in \mathrm{Aut}(L)$.

Proof. We can prove (exercise!) that $\frac{\delta^n}{n!}[x,y] = \sum_{i=0}^{n} \left[\frac{\delta^i x}{i!}, \frac{\delta^{n-i} y}{(n-i)!}\right]$.
Using this, we can show (exercise!) that $[\exp\delta(x), \exp\delta(y)] = \exp\delta\,[x,y]$. Conclude $\exp\delta \in \mathrm{End}(L)$.
The inverse of $\exp\delta$ is given by $\sum_{j=0}^{k-1}(1-\exp\delta)^j$ (check as an exercise!). ∎

4 Solvability and nilpotency of Lie algebras

4.1 Solvability

Let L be a Lie algebra. Define the **derived series** of L as follows:

$$D^0(L) = L$$

$$D^{n+1}(L) = [D^n(L), D^n(L)]$$

We say L is **solvable** if $D^k(L) = 0$ for some k.

Example (Derived series)**.** Consider $L = \mathfrak{t}(n,F)$, the Lie algebra of all (non-strict) upper triangular matrices, with the commutator $[AB] = AB - BA$. It is not hard to see that the diagonal elements of $AB - BA$ are all 0 whenever $A, B \in L$. It follows that $D^1(L) = [L,L] = \mathfrak{n}(n,F)$ the algebra of strictly upper triangular matrices.
In fact, for a matrix $A = (a_{ij}) \in L$ define $\min\{j - i : a_{ij} \neq 0\}$ to be the level of A. Denote the set of all matrices of level l by $\mathfrak{t}_l(n,F)$ and $\mathfrak{t}_k = 0 \forall k \geq n$. So $\mathfrak{t}_0 = \mathfrak{t}$ and $\mathfrak{t}_1 = \mathfrak{n}$. Turns out that $D^l(\mathfrak{t}) = \mathfrak{t}_{2^{l-1}}$ for $l \geq 1$. Note that these are all ideals of $\mathfrak{t}_0$.

Proposition 4.1.

1. All subalgebras and homomorphs of a solvable Lie algebra are solvable.
2. If I is a solvable ideal of a Lie algebra L such that L/I is solvable, then L is solvable.
3. Sum of solvable ideals of a Lie algebra is solvable.

Proof.

1. If L' is a subalgebra of L then $D^i(L') \subseteq D^i(L)$.
   If $\varphi : L \to L'$ is a surjective homomorphism, then $\varphi(D^i(L)) = D^i(L')$.
2. First note that $D^i(D^j(L)) = D^{i+j}(L)$. Consider the natural projection $\pi : L \twoheadrightarrow L/I$. Say k, l are such that $D^k(L/I) = 0, D^l(I) = 0$. By the second statement in the proof of Item 1, we have $\pi(D^k(L)) = D^k(L/I) = 0 \implies D^k(L) \subseteq \ker\pi = I \implies D^{k+l}(L) \subseteq D^l(I) = 0$.

3. By an isomorphism theorem, $(I+J)/J \cong I/(I\cap J)$. The RHS is the image of I under the natural projection $\varpi : I \twoheadrightarrow I/(I\cap J)$, and thus solvable by Item 1. Since J is solvable, conclude by Item 2 that $I+J$ is solvable.

■

Suppose S is a maximal solvable ideal of L, i.e., there is no solvable ideal properly containing S. Let $I \subseteq L$ be any solvable ideal. By the above proposition, $S+I$ is solvable. Further, it contains S. So $S = S+I$. It follows that $I \subseteq S$. This proves the following

Lemma 4.2. Every Lie algebra has a unique maximal solvable ideal.

The maximal solvable ideal of L is called the **radical of** L and denoted by $\operatorname{Rad} L$. L is said to be **semisimple** if $\operatorname{Rad} L = 0$. Clearly, L is semisimple iff L has no nonzero solvable ideals.

Example. Say L is simple, i.e., L has exactly two ideals, namely, 0 and L. Now $[L,L] = D^1(L)$ is an ideal of L, and nonzero (as L is non-abelian, by definition). This forces $D^k(L) = L \forall k$. The only solvable ideal of L is thus 0, which means L is semisimple.

Let's look at a slightly different characterization.

Proposition 4.3. L is semisimple iff L has no nonzero abelian ideals.

Proof. Say L is semisimple. If I is an abelian ideal of L, then $D^1(I) = 0$ whence I is solvable. It follows that $I = 0 \because I \subseteq \operatorname{Rad} L = 0$.
Say L has no nonzero abelian ideals. For any solvable ideal I, there is no nonzero term in the derived series, else the last nonzero term would be abelian. So $I = 0$. ■

Finally, an easy but important property of semisimple Lie algebras is

4.2 Nilpotency

Let L be a Lie algebra. Define the **central series** of L as follows:

$$C^0(L) = L$$

$$C^{n+1}(L) = [L, C^n(L)]$$

We say L is **nilpotent** if $C^k(L) = 0$ for some k.

Proposition 4.4. 1. All subalgebras and homomorphs of a nilpotent Lie algebra are nilpotent.

2. If $L/Z(L)$ is nilpotent, then L is nilpotent.

3. If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$

Proof. 1. Exactly as in solvability.

2. Consider the natural projection $\pi : L \twoheadrightarrow L/Z(L)$. Say k is such that $C^k(L/Z(L)) = 0$. Now $\pi(C^k(L)) = C^k(L/Z(L)) = 0 \implies C^k(L) \subseteq \ker\pi = Z(L) \implies C^{k+1}(L) \subseteq [L, Z(L)] = 0$.

3. Let k be least such that $C^k(L) = 0$. $L \neq 0 \implies k \geq 1$. Then $C^{k-1}(L) \neq 0$ and $\left[C^{k-1}(L), L\right] = 0 \implies 0 \neq C^{k-1}(L) \subseteq Z(L) \implies Z(L) \neq 0$.

■

It might be worthy to note that nilpotency implies solvability. We finally look at an example which is solvable but not nilpotent.

Example. Recall the two dimensional nonabelian Lie algebra: $L = Fx + Fy$ with $[x, y] = x$. Then $C^1 = D^1 = [L, L] = \langle [x, y] \rangle = Fx$. However, $D^2 = \left[D^1, D^1\right] = 0 \because$ 1-dimensional algebras are abelian and it follows that L is solvable. But $C^2 = \left[L, C^1\right] = Fx$. Consequently $C^k = Fx \forall k \geq 0$ whence L is not nilpotent.

Nilpotency is defined by looking at the central series. At an elemental level, we are really looking at terms $[x, y] = \mathrm{ad}_x(y)$ where $x \in L, y \in C^k(L)$. If L is nilpotent, we can conclude that there is some n for which $\mathrm{ad}_{x_n} \cdots \mathrm{ad}_{x_1}(y) = 0 \forall x_1, \cdots, x_n, y \in L$. In particular, $(\mathrm{ad}_x)^n = 0 \forall x \in L$. In other words,

Lemma 4.5. If L is nilpotent, then all elements of L are *ad*-nilpotent.

It turns out that the converse of the above lemma is also true and we call it Engel's theorem:

Theorem 4.6 (Engel)**.** If all elements of L are ad-nilpotent, then L is nilpotent.

Before proceeding with the proof (which is very very involved), we state (and not prove) an intermediate theorem, using which we will prove Theorem 4.6.

Theorem 4.7. Let $V \neq 0$ be a finite dimensional vector space and $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of only nilpotent elements. Then there is a common eigenvector for all elements of L. In other words, $\exists \vec{v} \in V, \vec{v} \neq 0$ such that $L\vec{v} = 0$.

Proof of Engel's theorem. Let L be a Lie algebra in which all elements are ad-nilpotent. So all elements of $\mathrm{ad}(L) = \{\mathrm{ad}_{\boldsymbol{x}} : \boldsymbol{x} \in L\} \subseteq \mathfrak{gl}(L)$ are nilpotent, by definition. It follows (by the above theorem) that $\exists \vec{v} \in L \setminus \{0\}$ such that $\mathrm{ad}_{\boldsymbol{x}}(\vec{v}) = 0 \forall \boldsymbol{x} \in L$, i.e., $[\vec{v}, L] = 0$. So $\vec{v} \in Z(L)$. Note that $L' = L/Z(L)$ is a Lie algebra of smaller dimension ($\because Z(L) \neq 0$) and all elements of L' are ad-nilpotent. By induction, that L' is nilpotent, whence by an earlier proposition, L is nilpotent. ■

4.3 A theorem of Lie

From here, we will assume $F = \overline{F}$.
Following up on Theorem 4.7, we have a similar theorem for solvable (instead of 'nilpotent') algebras, which we again state without proof.

Theorem 4.8. Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$ with $V \neq 0$ finite dimensional vector space. V contains a common eigenvector for all endomorphisms in L. In other words, there is a $\vec{v} \in L \setminus \{0\}$ and a functional $\lambda : L \to F$ such that $\boldsymbol{x}\vec{v} = \lambda(\boldsymbol{x})\vec{v} \forall \boldsymbol{x} \in L$.

Theorem 4.9 (Lie' theorem)**.** Let $V \neq 0$ be a finite dimensional vector space. Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then $\exists$ a flag $0 = V_0 \subseteq V_1 \subset \cdots \subset V_n = V$ (i.e., $\dim V_i = i$) and is stable under L (i.e., $[L, V_i] \subseteq V_i \forall i$).
(In other words, the matrices of L relative to a suitable basis of V are all upper triangular).

Proof. Let $\vec{v}$ be an eigenvector as stated in the previous theorem and let $V_1 = F\vec{v}$. Consider $\tilde{V} = V/V_1$. This forces us to consider the homomorphism $\tilde{\varphi} : L \to \mathfrak{gl}\left(\tilde{V}\right)$ given by $\tilde{\varphi}(\boldsymbol{x}) = (\vec{u} + V_1 \mapsto \boldsymbol{x}\vec{u} + V_1)$. By induction, there is a flag $0 = V_1/V_1 \subset V_2/V_1 \subset \cdots \subset V_n/V_1$ stable under $\tilde{\varphi}(L)$, for some subspaces $V_2 \subset \cdots \subset V_n$, and $\dim(V_i/V_1) = i - 1$. So we have found $V_2 \subset \cdots \subset V_n$ (take any lift) with dimensions $2, \cdots, n$ respectively, which are stable under L. ■

4.4 A theorem of Cartan

In a similar style, we state an intermediate theorem which will be useful in proving a theoreym due to Cartan, which gives a criterion for solvability.

Theorem 4.10. Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ with V being a finite dimensional vector space. Consider $M = \{x \in \mathfrak{gl}(V) : [x, B] \subseteq A\}$. Suppose $x \in M$ satisfies $\mathrm{Tr}(xy) = 0 \forall y \in M$. Then x is nilpotent.

Theorem 4.11 (Cartan's criterion)**.** Let L be a subalgebra of $\mathfrak{gl}(V)$, with V being a finite dimensional vector space. If $\mathrm{Tr}(xy) = 0 \forall x \in [LL], y \in L$ then L is solvable.

Proof. Use Theorem 4.10 with $A = [LL], B = L, V$ as given, along with the following associativity: $\mathrm{Tr}([xy]\,z) = \mathrm{Tr}(x\,[yz])$ if x, y, z are endomorphisms of some finite dimensional vector space. ■

5 Representations

Definition 5.1 (Representation)**.** Let L be a Lie algebra over field F. A **representation** of L is a homomorphism $\rho : L \to \mathfrak{gl}(V)$ along with some vector space V/F.

Definition 5.2 (Module)**.** Let L be a Lie algebra over field F. An ***L*-module** is a vector space V endowed with an operation $L \times V \to V$ (denoted $(\boldsymbol{x}, \vec{v}) \mapsto \boldsymbol{x}\vec{v}$) such that the following hold $\forall a, b \in F, \boldsymbol{x}, \boldsymbol{y} \in L, \vec{u}, \vec{v} \in V$:

1. $(a\boldsymbol{x} + b\boldsymbol{y})v = a(\boldsymbol{x}\vec{v}) + b(\boldsymbol{y}\vec{v})$
2. $\boldsymbol{x}(a\vec{u} + b\vec{v}) = a(\boldsymbol{x}\vec{u}) + b(\boldsymbol{x}\vec{v})$
3. $[\boldsymbol{x}\boldsymbol{y}]\,\vec{v} = \boldsymbol{x}\,(\boldsymbol{y}\vec{v}) - \boldsymbol{y}\,(\boldsymbol{x}\vec{v})$

These are exactly the same:

- For a representation $\rho : L \to \mathfrak{gl}(V)$, V is an L-module via the action $\boldsymbol{x}\vec{v} = \rho(\boldsymbol{x})\vec{v}$.
- If V is an L-module (action $(\boldsymbol{x}, \vec{v}) \mapsto \boldsymbol{x}\vec{v}$), then $\rho : L \to \mathfrak{gl}(V)$ is a representation by defining $\rho(\boldsymbol{x}) = (\vec{v} \mapsto \boldsymbol{x}\vec{v})$.

Definition 5.3 (Irreducible representation)**.** An L-module V is said to be **irreducible** iff V has precisely two L-submodules, namely 0 and V.

The complete opposite of the above concept is:

Definition 5.4 (Completely reducible representation)**.** An L-module V is said to be **completely reducible** iff every L-submodule W of V has a complement W' which is also an L-submodule of V.

An equivalent characterization is

Proposition 5.5. An L-module V is said to be **completely reducible** iff V can be written as a sum of irreducible L-submodules.

Theorem 5.6 (Schur's lemma)**.** Let V, W be irreducible representations of a Lie algebra L, everything over field F. Let $\psi : V \to W$ be a homomorphism of L-modules. Then we have

1. (F may not be algebraically closed) $\psi = 0$ or ψ is an isomorphism.
2. $\left(F = \overline{F}\right)$ For $V = W$, we have $\psi = \lambda \cdot I$ for some $\lambda \in F$.

Proof. 1. $\ker\psi$ is an L-submodule. So $\ker\psi = 0$ or $\ker\psi = V$.
$\ker\psi = 0 \implies \psi(V) \cong V$. $\psi(V)$ is a L-submodule of W. It follows that by irreducibility of W that $W \cong \psi(V) \cong V$.
$\ker\psi \neq 0 \implies \ker\psi = V \implies \psi = 0$.

2. $\psi \in \mathrm{End}(V)$. Let $\lambda \in F = \overline{F}$ be an eigenvalue of ψ. So $\ker(\psi - \lambda \cdot I) \neq 0 \implies \psi = \lambda \cdot I$ ■

Corollary 5.7. Let $\varphi : L \to \mathfrak{gl}(V)$ be irreducible. If $\psi \in \mathfrak{gl}(V)$ is such that $[\psi, \varphi_{\boldsymbol{x}}] = 0 \forall x \in L$ then ψ is a scalar.

Proof. Write $\varphi_{\boldsymbol{x}}(\vec{v})$ as $\boldsymbol{x}\cdot\vec{v}$. $[\psi, \varphi_{\boldsymbol{x}}] = 0$ simply means that $\psi(\boldsymbol{x}\cdot\vec{v}) = \boldsymbol{x}\cdot\psi(\vec{v})$, that is, $\psi \in \mathrm{End}_L(V)$. Conclude by Theorem 5.6. ■

Consider a symmetric bilinear form $\beta : L \times L \to F$.
Its radical is defined to be $S = \{x \in L : \beta(x, y) = 0 \forall L\} = \{x \in L : \beta(x, L) = 0\}$. If the radical is 0 we say β is **nondegenerate**.
A different way to see nondegeneracy is as follows. Fix a basis $\mathcal{B} = (e_1, \cdots, e_n)$ of L. β is nondegenerate iff the matrix $M = [\beta(e_i, e_j)]_{ij}$ is nonsingular.
This can be seen as follows: $S \neq 0 \iff \beta(x, L) = 0$ for some $x \neq 0 \iff \exists x \in L \setminus \{0\}$ such that $\beta(e_i, x) = 0 \forall i \iff M[x]_{\mathcal{B}} = 0$ for some $x \neq 0 \iff \det M = 0$.
Here is an interesting corollary to Theorem 5.6.

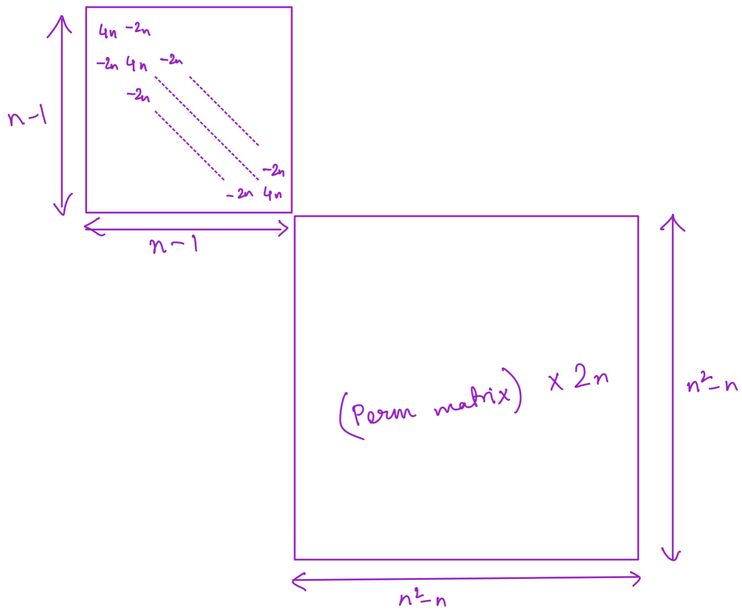
Corollary 5.8. $(F = \overline{F})$ Any two nondegenerate symmetric bilinear forms on a simple Lie algebra L are proportional.

Proof. Let β, γ be two such bilinear forms.
Define $\varphi : L \to L^*$ by $\varphi(x) = \beta_x = \beta(x, \cdot)$. φ_x is an L-module homomorphism $\forall x$. Now, $\forall f \in L^* \exists\, x_f \in L$ such that $f = \gamma(x_f, \cdot) \because \gamma$ is nondegenerate. Let $\sigma : L^* \to L$ be the map $f \mapsto x_f$. This is again a homomorphism.
One can show that (exercise!) $(\sigma \circ \varphi) \circ \mathrm{ad}_x(v) = \mathrm{ad}_x(\sigma \circ \varphi)(v) \forall x, v \in L$. In other words, $\sigma \circ \varphi \in \mathfrak{gl}(L)$ commutes with every ad_x. Note that $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is an irreducible representation due to the simplicity of L. By Corollary 5.7, $\sigma \circ \varphi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in F$. But this just means that $\beta(x, y) = \beta_x(y) = \gamma(\sigma(\beta_x), y) = \gamma(\sigma(\varphi(x)), y) = \gamma(\lambda x, y) = \lambda \cdot \gamma(x, y)$. ■

5.1 Casimir element of a representation and its decomposition

Let L be a semisimple Lie algebra along with a monomorphism $\varphi : L \to \mathfrak{gl}(V)$, for some finite dimensional vector space V, denoted by $x \mapsto \varphi_x$. In our language, such a monomorphism will also be called a **faithful** representation. Consider the bilinear map $\beta : L \times L \to F$ defined by $\beta(x, y) = \mathrm{Tr}(\varphi_x \varphi_y)$. It is not hard to see that β is a symmetric bilinear form. Further, β is associative in the following sense: $\beta([x, y], z) = \beta(x, [y, z])$.
Due to the faithfulness of the representation, β, in this case, turns out to be nondegenerate (hint: Cartan's criterion for solvability).

Example (A special case: $\varphi = \mathrm{ad}$). The β defined above (for any Lie algebra, not necessarily semisimple) corresponding to the adjoint representation is known as the **killing form** and denoted by κ. It turns out, due to a (different) criterion of Cartan that a Lie algebra is semisimple iff it's killing form is nondegenerate. On some thought (I found this by pure experimentation), it turns out that the killing form of $\mathfrak{sl}(n, F)$ looks like (here, the choice of basis is: $e_{11} - e_{22}, \cdots, e_{n-1,n-1} - e_{nn}, e_{12}, e_{13}, \cdots, e_{1n}, e_{21}, \cdots, e_{2n}, \cdots, e_{n-1,n}$):



We can compute the determinant of this as follows. The final answer would be the product of the determinant of the two blocks. The upper block has determinant $n^{n^2} \times 2^{n^2-1}$ (take $2n$ common and solve a recurrence). The bottom block is a (symmetric) permutation matrix of size $n^2 - n$, so it is a product of $\frac{n^2-n}{2} = \binom{n}{2}$ transpositions. The overall determinant turns out to be $(-1)^{\binom{n}{2}} n^{n^2} \times 2^{n^2-1}$. In other words, this is $\begin{cases} n^{n^2} \times 2^{n^2-1} & \text{if } n \equiv 0, 1 \pmod 4 \\ -n^{n^2} \times 2^{n^2-1} & \text{otherwise} \end{cases}$.

Back to our discussion. Let γ be any nondegenerate symmetric associative bilinear form on L. Consider the fixed basis $\mathcal{B} = (e_i)_{i=1}^n$ of L. And let $f_1, \cdots, f_n$ be a basis dual to $\mathcal{B}$ with respect to γ (this makes sense because γ is nondegenerate). For any representation $\rho : L \to \mathfrak{gl}(V)$ define $c_\rho(\gamma) = \sum_i \rho(e_i)\rho(f_i)$.

Definition 5.9. For a faithful representation $\varphi : L \to \mathfrak{gl}(V)$ denoted by $x \mapsto \varphi_x$, with trace form $\beta(x, y) = \mathrm{Tr}(\varphi_x \varphi_y)$, the map $c_\varphi(\beta)$ defined above is called the **Casimir element** of the representation φ with respect to the chosen bases. Since the information of β is encoded in φ, we simply call this c_φ.

Claim 5.9.1. For $x \in L$ define a_{ij}, b_{ij} to be such that $[x, e_i] = \sum_j a_{ij} e_j$ and $[x, f_i] = \sum_j b_{ij} f_j$. Then $a_{ij} + b_{ji} = 0 \forall 1 \le i, j \le n$.

Proof. Exercise. ■

With this claim, we can easily see that for any representation ρ and nondegenerate symmetric bilinear form γ, $[c_\rho(\gamma), \rho(x)] = 0 \forall x \in L$. Further, if V is irreducible, then $c_\rho = \frac{n}{k} I_k$, where $k = \dim V, n = \dim L$. This can be seen by Schur's lemma. It follows that such c_ρ is independent of the chosen basis.

Remark 5.10. It turns out that $L = [LL]$ for semisimple Lie algebras L. The proof of this statement requires some more machinery to be built up. Intuitively, 'semisimple' means something which is built out of 'simple' stuff. It is indeed true that every semisimple Lie algebra can be written as the direct sum of simple ideals of L. Further this decomposition is unique. Since $L = [LL]$ holds for simple Lie algebras, it will hold for their direct sum too, because, the 'components' are independent in some sense.

Now notice that for any representation $\varphi : L \to \mathfrak{gl}(V)$, we have $\varphi_x = \varphi_{[yz]}$ for some $y, z \in L$ due to Remark 5.10. This just means that $\varphi_x = [\varphi_y, \varphi_z] \in \mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$. Further we recall that a representation is said to be completely reducible if it can be written as a sum of irreducible representations. We end by stating important theorem, without proof, in light of the above discussion:

Theorem 5.11 (Weyl). Let $\varphi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra L. Then φ is completely reducible.