Real Analysis

Cantor Set

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1 Defining the Cantor set

For a set $S \subseteq \mathbb{R}$, we let $a \cdot S := \{ax : x \in S\}, a + S := \{a + x : x \in S\}$ for any $a \in \mathbb{R}$.
Start with $\mathscr{F}_0 := [0,1]$. Inductively define $\mathscr{F}_{k+1} := \left(\frac{1}{3}\mathscr{F}_k\right) \cup \left(\frac{2}{3} + \frac{1}{3}\mathscr{F}_k\right)$. For example, $\mathscr{F}_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $\mathscr{F}_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. By induction, $\mathscr{F}_k$ is a union of 2^k disjoint closed intervals. We define $\mathscr{F} := \bigcap_{k \in \mathbb{N}} \mathscr{F}_k$ to be the **Cantor set**. Note, $\mathscr{F}_k \supseteq \mathscr{F}_{k+1}$. So this is a decreasing sequence of nonempty compact sets, which means $\mathscr{F}$ is nonempty. In fact, this is compact (closed because intersection of closed sets, bounded because contained in $[0,1]$). We will eventually show that $\mathscr{F}$ is an uncountable set.

For now, note that $\mathscr{F}$ is closed. Take any $a, b \in \mathbb{R}$ with $a < b$. Take $m \in \mathbb{N}$ to be such that $3^m > \dfrac{6}{b-a}$. For such a choice of m, $\left(\dfrac{3k+1}{3^m}, \dfrac{3k+2}{3^m}\right) \subset (a,b)$. But, by the description of $\mathscr{F}$ is is not hard to see that $\left(\dfrac{3k+1}{3^m}, \dfrac{3k+2}{3^m}\right) \cap \mathscr{F} = \emptyset \forall k, m \geq 1$. It follows that $\mathscr{F}$ cannot contain any open ball, whence $F^o = \emptyset$. By definition, $\mathscr{F}$ is rare or nowhere dense.

2 Ternary expansions

Consider a sequence of numbers $\mathfrak{A} = (a_i)_{i \in \mathbb{N}}$ taking values in $\{0,1,2\}$. We define a rule $f(\mathfrak{A}) := \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$. Note that the sequence given by $S_n = \displaystyle\sum_{i=1}^{n} \frac{a_i}{3^i}$ is an increasing sequence. Further $S_n \leq \displaystyle\sum_{i=1}^{n} \frac{2}{3^i} \leq \frac{2}{3} \cdot \frac{1}{1-\frac{1}{3}} = 1 \forall n$. So, $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ is a real number in $[0,1]$. So $f : \{0,1,2\}^{\mathbb{N}} \to [0,1]$ is a well defined function. We will show that f is surjective but not injective.

Proposition 1 *f is not injective.*

Proof Consider the sequences $\mathfrak{A}_1 = (1,0,0,0,\cdots), \mathfrak{A}_2 = (0,2,2,2,\cdots)$. We note that $f(\mathfrak{A}_1) = \dfrac{1}{3}$ and that $f(\mathfrak{A}_2) = \displaystyle\sum_{i=2}^{\infty} \frac{2}{3} = \frac{2}{9} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{3}$. So we have found $\mathfrak{A}_1 \neq \mathfrak{A}_2$ with $f(\mathfrak{A}_1) = f(\mathfrak{A}_2)$. ■

We do a somewhat more general analysis and determine exactly what are the cases when curious things (as above happen). That is, we ask that if two sequences $\mathfrak{A} = (a_n), \mathfrak{B} = (b_n)$ satisfy that $f(\mathfrak{A}) = f(\mathfrak{B})$, then what are the conditions on $\mathfrak{A}, \mathfrak{B}$.
So we are assuming that $\displaystyle\sum_{i=1}^{\infty} \frac{a_i}{3^i} = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$ and that $\mathfrak{A} \neq \mathfrak{B}$. So $\exists k \in \mathbb{N}$ such that $a_k \neq b_k$, and take k to be the least

such. WLOG assume $a_k > b_k$. Now $\sum_{i=1}^{\infty}\frac{a_i}{3^i} = \sum_{i=1}^{\infty}\frac{b_i}{3^i} \implies \sum_{i=k}^{\infty}\frac{a_i}{3^i} = \sum_{i=k}^{\infty}\frac{b_i}{3^i} \implies \frac{a_k-b_k}{3^k} + \sum_{i=k+1}^{\infty}\frac{a_i}{3^i} = \sum_{i=k+1}^{\infty}\frac{b_i}{3^i}$. Note $\frac{1}{3^k} \le \frac{1}{3^k} + \sum_{i=k+1}^{\infty}\frac{a_i}{3^i} \le \frac{a_k-b_k}{3^k} + \sum_{i=k+1}^{\infty}\frac{a_i}{3^i} = \sum_{i=k+1}^{\infty}\frac{b_i}{3^i} \le \frac{2}{3^{k+1}}\cdot\frac{3}{2} = \frac{1}{3^k}$. This means $(a_i, b_i) = (0,2)\forall i > k, a_k - b_k = 1$. So the only 'curious' cases is one of the following two types:

$$0.s_1s_2\cdots s_m1000\cdots = 0.s_1s_2\cdots s_m0222\cdots \qquad 0.s_1s_2\cdots s_m2000\cdots = 0.s_1s_2\cdots s_m1222\cdots$$

But the set of these numbers is just the set of all numbers of the form $\frac{t}{3^k}$.

Now define a function $g : [0,1] \to \{0,1,2\}^{\mathbb{N}}$. First we say that if $x = \frac{t}{3^k}$ for some integers $t, k \ge 0$ we take the ternary expansion which has lesser usage of 1's.
Now for any other $x \in [0,1]$, indutively define a sequence $\mathfrak{A} = (a_n) \in \{0,1,2\}^{\mathbb{N}}$ as follows: Let a_1 be largest so that $\frac{a_1}{3} \le x$; and we let a_{m+1} to be the largest so that $\frac{a_{m+1}}{3^{m+1}} \le x - \sum_{i=1}^{m}\frac{a_i}{3^i}$. By induction, it follows that $0 \le x - \sum_{i=1}^{m}\frac{a_i}{3^i} < \frac{1}{3^m}$. This gives a sequence $\mathfrak{A}$ such that $\sum_{n=1}^{\infty}\frac{a_n}{3^n} = x$. It's not hard to see that $f(g(x)) = x\forall x \in [0,1]$. In other words, we have proved the

Proposition 2 *f is surjective.*

3 Relation between $\mathscr{F}$ and ternary expansion

From now on, whenever we say 'the ternary expansion of x' we always mean $\sum_{n=1}^{\infty}\frac{a_n}{3^n}$ where $(a_n) = g(x)$. And we will mean $\{1,\cdots,n\}$ when we write $[n]$. Also, for a sequence $\mathfrak{A} = (a_n)$ we define the i^{th} projection map as $\pi_i(\mathfrak{A}) := a_i$. Let $\mathscr{G}_1 := \{x \in [0,1] : \pi_j(g(x)) \ne 1 \forall j \in [1]\}$, that is, the set of all $x \in [0,1]$ such that the first term in its ternary expansion is not 1. It is not hard to see that $\mathscr{G}_1 = \mathscr{F}_1$. Indeed, $\pi_1(g(x)) = 1 \iff x \in \left(\frac{1}{3}, \frac{2}{3}\right)$. Similarly define $\mathscr{G}_2 := \{x \in [0,1] : \pi_j(g(x)) \ne 1 \forall j \in [2]\}$ and observe that $\mathscr{G}_2 = \mathscr{F}_2$. In fact, it is true that $\mathscr{G}_n = \mathscr{F}_n \forall n \in \mathbb{N}$ where $\mathscr{G}_n := \{x \in [0,1] : \pi_j(g(x)) \ne 1 \forall j \in [n]\}$. It is thus clear that $\mathscr{G} := \bigcap_{k\in\mathbb{N}}\mathscr{G}_k = \bigcap_{k\in\mathbb{N}}\mathscr{F}_k = F$. But, $\mathscr{G} = f\left(\{0,2\}^{\mathbb{N}}\right)$. By our earlier discussion, we have seen exactly when f fails to be injective. In particular $f|_{\{0,2\}^{\mathbb{N}}}$ is injective. Uncountability of $\{0,2\}^{\mathbb{N}}$ implies the uncountability of $\mathscr{F}$.

Corollary 3 *$\forall r > 0, a \in \mathscr{F}, \exists b \in \mathscr{F}$ such that $0 < |b-a| < r$. In other words, $\mathscr{F}$ has no isolated point.*

Proof Let $r > 0, a \in \mathscr{F}$. Take $m \in \mathbb{N}$ to be such that $3^m > \frac{2}{r}$. Say $g(a) = (a_n)$.
Define $\mathfrak{B} := (a_1, \cdots, a_{m-1}, 2 - a_m, a_{m+1}, a_{m+2}, \cdots)$ and $b := f(\mathfrak{B})$. Clearly $|b-a| = \frac{2}{3^m} \in (0, r)$. ■

Finally, we exhibit a surjection $\mathscr{F} \to [0,1]$. Note that $\tilde{g} := g|_{\mathscr{G}} = g|_{\mathscr{F}}$ is a surjection whose image is $\{0,2\}^{\mathbb{N}}$. Next define $h : \{0,2\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by $(a_n) \overset{h}{\mapsto} \left(\frac{\pi_n(a_n)}{2}\right)$. h is surjective as well. Lastly, notice that the map $\rho : \{0,1\}^{\mathbb{N}} \to [0,1]$ given by $(a_n) \mapsto \sum_{n=1}^{\infty}\frac{a_n}{2^n}$ is well defined and a surjection (by the same argument used to prove proposition 2). The surjectivity of all these maps proves the surjectivity of $(\rho \circ h \circ \tilde{g}) : \mathscr{F} \to [0,1]$.
In short, if the ternary expansion of $x \in \mathscr{F}$ is $\sum_{n=1}^{\infty}\frac{a_n}{3^n}$ (so that each $a_n \in \{0,2\}$) then we map it to $\sum_{n=1}^{\infty}\frac{a_n/2}{2^n}$. This is well defined because the ternary representation of elements of $\mathscr{F}$ is unique.